

On 2D Euler Equations: Part I. On the Energy-Casimir Stabilities and The Spectra for Linearized 2D Euler Equations

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Abstract

In this paper, we study a linearized two-dimensional Euler equation. This equation decouples into infinitely many invariant subsystems. Each invariant subsystem is shown to be a linear Hamiltonian system of infinite dimensions. Another important invariant besides the Hamiltonian for each invariant subsystem is found, and is utilized to prove an “unstable disk theorem” through a simple Energy-Casimir argument [1]. The eigenvalues of the linear Hamiltonian system are of four types: real pairs $(c, -c)$, purely imaginary pairs $(id, -id)$, quadruples $(\pm c \pm id)$, and zero eigenvalues. The eigenvalues are computed through continued fractions. The spectral equation for each invariant subsystem is a Poincaré-type difference equation, i.e. it can be represented as the spectral equation of an infinite matrix operator, and the infinite matrix operator is a sum of a constant-coefficient infinite matrix operator and a compact infinite matrix operator. We have obtained a complete spectral theory.

I. Introduction

In [2], Henshaw and Kreiss numerically studied the propagation of perturbations in the solution of the two dimensional incompressible Navier-Stokes equations with no body force, and at high Reynolds numbers. They numerically solved the equations starting with some smooth initial data for which the Fourier modes have random phases. This flow evolved over time, initially through a complicated state containing many shear layers and then into a state of large vortex structures (with shear layers) which persists for long time.

1. Changing the viscosity, the large vortex structures do not change much.
2. Adding high mode perturbation to the initial data, the large vortex structures do not change much.
3. Changing the Laplacian Δ to Δ^2 , the large vortex structures do not change much.
4. Changing the Laplacian Δ to an operator which is Δ for low modes, and is Δ^2 for high modes, the large vortex structures do not change much.

In [3], Matthaeus et al. studied the same problem, and found similar results. Matthaeus et al. run the numerics for much longer time. These numerics indicate that for relatively long time (not infinite long time), the solution to the 2D N-S equation (without body force, i.e. decaying turbulence) has the large vortex structures. Such structures persist in the solution to the 2D Euler equation by the claims 1, 3, 4 above. (Cf: Under decay boundary conditions, the Kato's theorem states that for finite time the solution to 2D N-S equation converges to the solution to 2D Euler equation in norm as viscosity approaches zero, see [4].) Moreover, the large vortex structures are stable with respect to the change in initial data.

In [5], Robert and Sommeria studied the organized structures in two-dimensional Euler fluid flows by a theory of equilibrium statistical mechanics. The theory takes into account all the known constants of motion for the two-dimensional Euler equations. The microscopic states are all the possible vorticity fields, while a macroscopic state is defined as a probability distribution of vorticity at each point of the domain, which describes in a statistical sense the fine-scale vorticity fluctuations. The organized structure appears as a state of maximal entropy, with the constraints of all the constants of motion. The vorticity field obtained as the local average of this optimal macrostate is a *steady solution* of the Euler equations.

The above numerical results show that certain relatively long time large vortex structures for 2D Euler equation persist for 2D N-S equation at high Reynolds numbers. Such structures are stable with respect to the change in initial data. We believe such structures are the exhibition of certain unstable manifolds. The above theoretical results show that the probability mean of such structures for 2D Euler flows are steady solutions to the 2D Euler equations. Thus, we believe that certain unstable manifolds, if there is any, of steady solutions to 2D Euler equations are responsible for such large vortex structures. Therefore, a dynamical system study on certain unstable manifolds of certain steady solutions to 2D Euler equations, and their persistence for 2D N-S equations at high Reynolds numbers is crucial for studying the large vortex structures. Indeed, we have built a *chaos-molecules-model* on 2D turbulence based upon the above motivations [6]. We believe that our chaos-molecules-model captures the qualitative frames of the hyperbolic structures, and the energy inverse cascade and the enstrophy cascade nature for 2D turbulence.

In this paper, we study the linearized 2D Euler equation at a fixed point. This study is the base for future analytical studies on the unstable manifolds for the 2D Euler equation.

In [7], we have begun numerical studies on the unstable manifolds for the 2D Euler equation.

The current study is also important in the linear hydrodynamic stability theory. By utilizing Energy-Casimir method [1], we obtain an *unstable disk theorem* which is not in the category of the classical Rayleigh theorem [8].

Next we discuss the approaches used in this study. Through the Energy-Casimir method, nonlinear stabilities of various types of two-dimensional ideal fluid flows have been established [9] [10] [11] [1]. Below we give a brief description on the Energy-Casimir method. Let D be a region on the (x, y) -plane bounded by the curves Γ_i ($i = 1, 2$), an ideal fluid flow in D is governed by the 2D Euler equation written in the stream-function form:

$$\frac{\partial}{\partial t} \Delta \psi = [\nabla \psi, \nabla \Delta \psi] , \quad (\text{I.1})$$

where

$$[\nabla \psi, \nabla \Delta \psi] = \frac{\partial \psi}{\partial x} \frac{\partial \Delta \psi}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \Delta \psi}{\partial x} ,$$

with the boundary conditions,

$$\psi|_{\Gamma_i} = c_i(t) , \quad c_1 \equiv 0 , \quad \frac{d}{dt} \oint_{\Gamma_i} \frac{\partial \psi}{\partial n} ds = 0 .$$

For every function $f(z)$, the functional

$$F = \int \int_D f(\Delta \psi) \, dx dy \quad (\text{I.2})$$

is a constant of motion (a Casimir) for (I.1). The conditional extremum of the kinetic energy

$$E = \frac{1}{2} \int \int_D \nabla \psi \cdot \nabla \psi \, dx dy \quad (\text{I.3})$$

for fixed F is given by the Lagrange's formula [9],

$$\delta H = \delta(E + \lambda F) = 0 , \quad \Rightarrow \quad \psi_0 = \lambda f'(\Delta \psi_0) . \quad (\text{I.4})$$

where λ is the Lagrange multiplier. Thus, ψ_0 is the stream function of a stationary flow, which satisfies

$$\psi_0 = \Phi(\Delta\psi_0) , \quad (\text{I.5})$$

where $\Phi = \lambda f'$. The second variation is given by [9],

$$\delta^2 H = \frac{1}{2} \int \int_D \left\{ \nabla \phi \cdot \nabla \phi + \Phi'(\Delta\psi_0) (\Delta\phi)^2 \right\} dx dy . \quad (\text{I.6})$$

Let $\psi = \psi_0 + \varphi$ be a solution to the 2D Euler equation (I.1), Arnold proved the estimates [10]: (a). when $c \leq \Phi'(\Delta\psi_0) \leq C$, $0 < c \leq C < \infty$,

$$\int \int_D \left\{ \nabla \varphi(t) \cdot \nabla \varphi(t) + c[\Delta\varphi(t)]^2 \right\} dx dy \leq \int \int_D \left\{ \nabla \varphi(0) \cdot \nabla \varphi(0) + C[\Delta\varphi(0)]^2 \right\} dx dy,$$

for all $t \in (-\infty, +\infty)$, (b). when $c \leq -\Phi'(\Delta\psi_0) \leq C$, $0 < c < C < \infty$,

$$\int \int_D \left\{ c[\Delta\varphi(t)]^2 - \nabla \varphi(t) \cdot \nabla \varphi(t) \right\} dx dy \leq \int \int_D \left\{ C[\Delta\varphi(0)]^2 - \nabla \varphi(0) \cdot \nabla \varphi(0) \right\} dx dy,$$

for all $t \in (-\infty, +\infty)$. Therefore, when the second variation (I.6) is positive definite, or when

$$\int \int_D \left\{ \nabla \phi \cdot \nabla \phi + [\max \Phi'(\Delta\psi_0)] (\Delta\phi)^2 \right\} dx dy$$

is negative definite, the stationary flow (I.5) is nonlinearly stable (Liapunov stable). In this paper, we have found an invariant for the linearized 2D Euler equation, and use this invariant together with an Energy-Casimir type argument to study linear stability, and to prove an unstable disk theorem. The linearized 2D Euler equation is an infinite dimensional linear Hamiltonian system. For finite dimensional linear Hamiltonian systems, it is well-known that the eigenvalues are of four types: real pairs $(c, -c)$, purely imaginary pairs $(id, -id)$, quadruples $(\pm c \pm id)$, and zero eigenvalues [12] [13] [14]. The same is true for the linearized 2D Euler equation. The eigenvalues are computed through continued fractions following the

work of Meshalkin and Sinai [15]. The linearized 2D Euler equation can also be written in an infinite matrix form. The spectral equation of the infinite matrix operator defines a Poincaré-type difference equation [16] [17]. That is, the infinite matrix operator can be written as the sum of a constant-coefficient infinite matrix operator and a compact infinite matrix operator. In this paper, we follow a spectral theory developed by Duren [18] to study the spectra of the constant-coefficient infinite matrix operator through characteristic polynomials. Then we apply the Weyl's essential spectra theorem to the perturbation of the constant-coefficient infinite matrix operator by the compact infinite matrix operator [19], to achieve a complete spectral theory.

Finally, we discuss some preliminaries on the 2D Euler equation. Consider the two-dimensional incompressible Euler equation written in vorticity form,

$$\frac{\partial \Omega}{\partial t} = -u \frac{\partial \Omega}{\partial x} - v \frac{\partial \Omega}{\partial y}, \tag{I.7}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0;$$

under periodic boundary conditions in both x and y directions with period 2π , where Ω is vorticity, u and v are respectively velocity components along x and y directions. We also require that both u and v have means zero,

$$\int_0^{2\pi} \int_0^{2\pi} u \, dx dy = \int_0^{2\pi} \int_0^{2\pi} v \, dx dy = 0.$$

Expand Ω into Fourier series,

$$\Omega = \sum_{k \in \mathbb{Z}^2 / \{0\}} \omega_k e^{ik \cdot X},$$

where $\omega_{-k} = \overline{\omega_k}$, $k = (k_1, k_2)^T$, $X = (x, y)^T$. In this paper, we confuse 0 with $(0, 0)^T$, the context will always make it clear. By the relation between vorticity Ω and stream function

Ψ ,

$$\Omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \Delta \Psi ,$$

where the stream function Ψ is defined by,

$$u = -\frac{\partial \Psi}{\partial y} , \quad v = \frac{\partial \Psi}{\partial x} ;$$

the system (I.7) can be rewritten as the following kinetic system,

$$\dot{\omega}_k = \sum_{k=p+q} A(p, q) \omega_p \omega_q , \quad (\text{I.8})$$

where $A(p, q)$ is given by,

$$\begin{aligned} A(p, q) &= \frac{1}{2} [|q|^{-2} - |p|^{-2}] (p_1 q_2 - p_2 q_1) \\ &= \frac{1}{2} [|q|^{-2} - |p|^{-2}] \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} , \end{aligned} \quad (\text{I.9})$$

where $|q|^2 = q_1^2 + q_2^2$ for $q = (q_1, q_2)^T$, similarly for p .

Remark I.1 Notice that direct calculation shows that the nonlinear term in (I.8) is $\sum_{k=p+q} \tilde{A}(p, q) \omega_p \omega_q$, where $\tilde{A}(p, q) = |p|^{-2}(q_1 p_2 - p_1 q_2)$; then, $\tilde{A}(q, p) = |q|^{-2}(p_1 q_2 - q_1 p_2)$. The $A(p, q)$ in (I.8) is the average of $\tilde{A}(p, q)$ and $\tilde{A}(q, p)$, $A(p, q) = \frac{1}{2}[\tilde{A}(p, q) + \tilde{A}(q, p)]$.

For any two functionals F_1 and F_2 of $\{\omega_k\}$, define their Lie-Poisson bracket:

$$\{F_1, F_2\} = \sum_{k+p+q=0} \begin{vmatrix} q_1 & p_1 \\ q_2 & p_2 \end{vmatrix} \omega_k \frac{\partial F_1}{\partial \omega_p} \frac{\partial F_2}{\partial \omega_q} . \quad (\text{I.10})$$

Then the 2D Euler equation (I.8) is a Hamiltonian system [20],

$$\dot{\omega}_k = \{\omega_k, H\}, \quad (\text{I.11})$$

where the Hamiltonian H is the kinetic energy,

$$H = \frac{1}{2} \sum_{k \in Z^2/\{0\}} |k|^{-2} |\omega_k|^2. \quad (\text{I.12})$$

Following are Casimirs (i.e. invariants that Poisson commute with any functional) of the Hamiltonian system (I.11):

$$J_n = \sum_{k_1 + \dots + k_n = 0} \omega_{k_1} \dots \omega_{k_n}. \quad (\text{I.13})$$

The following Proposition is concerned with the equilibrium manifolds of the 2D Euler equation (I.8).

Proposition 1 *For any $k \in Z^2/\{0\}$, the infinite dimensional space*

$$E_k^1 \equiv \left\{ \{\omega_{k'}\} \mid \omega_{k'} = 0, \text{ if } k' \neq rk, \forall r \in R \right\},$$

and the finite dimensional space

$$E_k^2 \equiv \left\{ \{\omega_{k'}\} \mid \omega_{k'} = 0, \text{ if } |k'| \neq |k| \right\},$$

entirely consist of fixed points of the system (I.8).

Proof: Let $k^0 \in Z^2/\{0\}$, $\{\omega_k^0\} \in E_{k^0}^1$. For any $p, q \in Z^2/\{0\}$, $\omega_p^0 \omega_q^0 \neq 0$ implies that $p = r_1 k^0$, $q = r_2 k^0$ for some $r_1, r_2 \in R$; then, $A(p, q) = 0$; thus we always have $A(p, q) \omega_p^0 \omega_q^0 = 0$, and $\{\omega_k^0\}$ is a fixed point of (I.8). Let $k^0 \in Z^2/\{0\}$, $\{\omega_k^0\} \in E_{k^0}^2$. For any $p, q \in Z^2/\{0\}$, $\omega_p^0 \omega_q^0 \neq 0$ implies that $|p| = |k^0|$, $|q| = |k^0|$; then, $A(p, q) = 0$; thus we always have $A(p, q) \omega_p^0 \omega_q^0 = 0$, and $\{\omega_k^0\}$ is a fixed point of (I.8). \square

Fig.1 shows an example on the locations of the modes ($k' = rk$) and ($|k'| = |k|$) in the definitions of E_k^1 and E_k^2 (Proposition 1).

The paper is organized as follows: Section II is on the formulations of the problem. Section III is on the Liapunov stability. Section IV is on the properties of the eigenvalues

of the linearized 2D Euler equation as a linear Hamiltonian system. Section V is on the continued fraction study of the eigenvalues. Section VI is on the infinite-matrix study of the spectra of the linearized 2D Euler equation. Section VII is the conclusion.

II. The Formulations of the Problem

Denote $\{\omega_k\}_{k \in Z^2/\{0\}}$ by ω . Consider the simple fixed point ω^* :

$$\omega_p^* = \Gamma, \quad \omega_k^* = 0, \text{ if } k \neq p \text{ or } -p, \quad (\text{II.1})$$

of the 2D Euler equation (I.8), which belongs to the two-dimensional intersection space $E_p^1 \cap E_p^2$ (Proposition 1), where Γ is an arbitrary complex constant. The *linearized two-dimensional Euler equation* at ω^* is given by,

$$\dot{\omega}_k = A(p, k-p) \Gamma \omega_{k-p} + A(-p, k+p) \bar{\Gamma} \omega_{k+p}. \quad (\text{II.2})$$

This is the linearized two-dimensional Euler equation that we are going to study in this paper.

Definition 1 (Classes) For any $\hat{k} \in Z^2/\{0\}$, we define the class $\Sigma_{\hat{k}}$ to be the subset of $Z^2/\{0\}$:

$$\Sigma_{\hat{k}} = \left\{ \hat{k} + np \in Z^2/\{0\} \mid n \in Z, \text{ } p \text{ is specified in (II.1)} \right\}.$$

See Fig.2 for an illustration of the classes. According to the classification defined in Definition 1, the linearized two-dimensional Euler equation (II.2) decouples into infinite many invariant subsystems:

$$\begin{aligned} \dot{\omega}_{\hat{k}+np} &= A(p, \hat{k} + (n-1)p) \Gamma \omega_{\hat{k}+(n-1)p} \\ &+ A(-p, \hat{k} + (n+1)p) \bar{\Gamma} \omega_{\hat{k}+(n+1)p}. \end{aligned} \quad (\text{II.3})$$

Each invariant subsystem can be rewritten as a linear Hamiltonian system as shown below.

Definition 2 (The Quadratic Hamiltonian) *The quadratic Hamiltonian $\mathcal{H}_{\hat{k}}$ is defined as:*

$$\mathcal{H}_{\hat{k}} = -2 \operatorname{Im} \left\{ \sum_{n \in Z} \rho_n \Gamma A(p, \hat{k} + (n-1)p) \omega_{\hat{k}+(n-1)p} \bar{\omega}_{\hat{k}+np} \right\} \quad (\text{II.4})$$

$$= - \left| \begin{pmatrix} p_1 & \hat{k}_1 \\ p_2 & \hat{k}_2 \end{pmatrix} \right| \operatorname{Im} \left\{ \sum_{n \in Z} \Gamma \rho_n \rho_{n-1} \omega_{\hat{k}+(n-1)p} \bar{\omega}_{\hat{k}+np} \right\},$$

where $\rho_n = [| \hat{k} + np |^{-2} - |p|^{-2}]$, “Im ” denotes “imaginary part ”.

Then the invariant subsystem (II.3) can be rewritten as a linear Hamiltonian system,

$$i \dot{\omega}_{\hat{k}+np} = \rho_n^{-1} \frac{\partial \mathcal{H}_{\hat{k}}}{\partial \bar{\omega}_{\hat{k}+np}}. \quad (\text{II.5})$$

Let

$$\omega_{\hat{k}+np} = \alpha_n + i\beta_n, \quad n \in Z,$$

i.e. α_n and β_n are the real and imaginary parts of $\omega_{\hat{k}+np}$. Then the linear Hamiltonian system (II.5) can be rewritten in the form,

$$\begin{cases} \dot{\alpha}_n &= \frac{1}{2} \rho_n^{-1} \frac{\partial \mathcal{H}_{\hat{k}}}{\partial \beta_n}, \\ \dot{\beta}_n &= -\frac{1}{2} \rho_n^{-1} \frac{\partial \mathcal{H}_{\hat{k}}}{\partial \alpha_n}, \end{cases} \quad (\text{II.6})$$

where

$$\begin{aligned} \mathcal{H}_{\hat{k}} &= \left| \begin{pmatrix} \hat{k}_1 & p_1 \\ \hat{k}_2 & p_2 \end{pmatrix} \right| \sum_{n \in Z} \rho_n \rho_{n-1} \left[\Gamma_r (\alpha_n \beta_{n-1} - \alpha_{n-1} \beta_n) \right. \\ &\quad \left. + \Gamma_i (\alpha_{n-1} \alpha_n + \beta_{n-1} \beta_n) \right], \end{aligned}$$

where $\Gamma = \Gamma_r + i \Gamma_i$, i.e. Γ_r and Γ_i are the real and imaginary parts of Γ . Explicitly,

$$\left\{ \begin{array}{l} \dot{\alpha}_n = \frac{1}{2} \left| \begin{array}{cc} \hat{k}_1 & p_1 \\ \hat{k}_2 & p_2 \end{array} \right| \left[\rho_{n+1} \Gamma_r \alpha_{n+1} - \rho_{n-1} \Gamma_r \alpha_{n-1} \right. \\ \quad \left. + \rho_{n+1} \Gamma_i \beta_{n+1} + \rho_{n-1} \Gamma_i \beta_{n-1} \right], \\ \dot{\beta}_n = -\frac{1}{2} \left| \begin{array}{cc} \hat{k}_1 & p_1 \\ \hat{k}_2 & p_2 \end{array} \right| \left[\rho_{n-1} \Gamma_r \beta_{n-1} - \rho_{n+1} \Gamma_r \beta_{n+1} \right. \\ \quad \left. + \rho_{n+1} \Gamma_i \alpha_{n+1} + \rho_{n-1} \Gamma_i \alpha_{n-1} \right]. \end{array} \right. \quad (\text{II.7})$$

If we rescale the variables as follows,

$$\tau = \frac{1}{2} \left| \begin{array}{cc} \hat{k}_1 & p_1 \\ \hat{k}_2 & p_2 \end{array} \right| t, \quad \tilde{\alpha}_n = \rho_n \alpha_n, \quad \tilde{\beta}_n = \rho_n \beta_n,$$

The linear Hamiltonian system (II.7) can be rewritten in the simpler form,

$$\left\{ \begin{array}{l} d\tilde{\alpha}_n/d\tau = \rho_n \left[\Gamma_r (\tilde{\alpha}_{n+1} - \tilde{\alpha}_{n-1}) + \Gamma_i (\tilde{\beta}_{n+1} + \tilde{\beta}_{n-1}) \right], \\ d\tilde{\beta}_n/d\tau = -\rho_n \left[\Gamma_r (\tilde{\beta}_{n-1} - \tilde{\beta}_{n+1}) + \Gamma_i (\tilde{\alpha}_{n+1} + \tilde{\alpha}_{n-1}) \right]. \end{array} \right.$$

Next we discuss the fixed point ω^* from a variational-principle point of view. Consider the kinetic energy and the enstrophy of the 2D Euler equation (I.8),

$$E = \frac{1}{2} \sum_{k \in Z^2/\{0\}} |k|^{-2} |\omega_k|^2, \quad J = \sum_{k \in Z^2/\{0\}} |\omega_k|^2.$$

If we extremize the kinetic energy E for fixed enstrophy $J = c$ (a constant), we have the critical states by the Lagrange formula,

$$\left\{ \begin{array}{l} \frac{\partial L}{\partial \lambda} = J - c = 0, \\ \frac{\partial L}{\partial \omega_k} = (|k|^{-2} + \lambda) \bar{\omega}_k = 0, \quad \forall k \in Z^2/\{0\}, \end{array} \right.$$

where $L = 2E + \lambda(J - c)$. Thus, the critical states satisfy the relations,

$$\left\{ \begin{array}{l} \lambda = -|q|^{-2}, \quad \text{for some } q \in Z^2/\{0\}, \\ \omega_k = 0, \quad \text{if } |k| \neq |q|, \\ \sum_{|k|=|q|} |\omega_k|^2 = c. \end{array} \right. \quad (\text{II.8})$$

Thus, we have the *critical manifold* for extremizing the kinetic energy for fixed enstrophy,

$$M_{|q|}^c = \left\{ \omega \mid \omega_k = 0, \text{ if } |k| \neq |q|, \sum_{|k|=|q|} |\omega_k|^2 = c \right\},$$

which is a submanifold of the equilibrium manifold E_q^2 (Proposition 1). Denote by I the linear combination,

$$I = 2E - |p|^{-2} J. \quad (\text{II.9})$$

Then, we have the following variational principle for the fixed point ω^* (II.1).

Variational Principle: The fixed point ω^* is a conditionally critical state of the kinetic energy E for fixed enstrophy $J = |\Gamma|^2$, and is an absolute critical state of I .

III. Liapunov Stability

Definition 3 (An Important Functional) For each invariant subsystem (II.3), we define the functional $I_{\hat{k}}$ which is the restriction of the functional I (II.9) to the class $\Sigma_{\hat{k}}$,

$$I_{\hat{k}} = I_{(\text{restricted to } \Sigma_{\hat{k}})} \quad (\text{III.1})$$

$$= \sum_{n \in \mathbb{Z}} \{ |\hat{k} + np|^{-2} - |p|^{-2} \} \left| \omega_{\hat{k}+np} \right|^2.$$

Lemma III.1 $I_{\hat{k}}$ is a constant of motion for the system (II.5).

Proof: Differentiating $I_{\hat{k}}$, we have

$$\begin{aligned} \dot{I}_{\hat{k}} &= \sum_{n \in \mathbb{Z}} \rho_n \left[\dot{\omega}_{\hat{k}+np} \bar{\omega}_{\hat{k}+np} + \omega_{\hat{k}+np} \dot{\bar{\omega}}_{\hat{k}+np} \right] \\ &= -i \sum_{n \in \mathbb{Z}} \left[\bar{\omega}_{\hat{k}+np} \frac{\partial \mathcal{H}_{\hat{k}}}{\partial \bar{\omega}_{\hat{k}+np}} - \omega_{\hat{k}+np} \frac{\partial \mathcal{H}_{\hat{k}}}{\partial \omega_{\hat{k}+np}} \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \left| \begin{array}{cc} p_1 & \hat{k}_1 \\ p_2 & \hat{k}_2 \end{array} \right| \sum_{n \in Z} \left[\bar{\omega}_{\hat{k}+np} \left(\Gamma \rho_n \rho_{n-1} \omega_{\hat{k}+(n-1)p} \right. \right. \\
&\quad \left. \left. - \bar{\Gamma} \rho_{n+1} \rho_n \omega_{\hat{k}+(n+1)p} \right) - \omega_{\hat{k}+np} \left(\Gamma \rho_{n+1} \rho_n \bar{\omega}_{\hat{k}+(n+1)p} \right. \right. \\
&\quad \left. \left. - \bar{\Gamma} \rho_n \rho_{n-1} \bar{\omega}_{\hat{k}+(n-1)p} \right) \right] = 0.
\end{aligned}$$

This completes the proof of the lemma. \square

Next we define the concept of disk in $Z^2/\{0\}$ which is needed in the unstable disk theorem to be proved below.

Definition 4 (The Disk) *The disk of radius $|p|$ in $Z^2/\{0\}$, denoted by $D_{|p|}$, is defined as*

$$D_{|p|} = \left\{ k \in Z^2/\{0\} \mid |k| < |p| \right\}.$$

The closure of $D_{|p|}$, denoted by $\bar{D}_{|p|}$, is defined as

$$\bar{D}_{|p|} = \left\{ k \in Z^2/\{0\} \mid |k| \leq |p| \right\}.$$

See Fig.2 for an illustration. Next we prove the unstable disk theorem using a simple Energy-Casimir type argument [20] [1].

Theorem III.1 (Unstable Disk Theorem) *If $\Sigma_{\hat{k}} \cap \bar{D}_{|p|} = \emptyset$, then the invariant subsystem (II.3) is Liapunov stable for all $t \in R$, in fact,*

$$\sum_{n \in Z} \left| \omega_{\hat{k}+np}(t) \right|^2 \leq \sigma \sum_{n \in Z} \left| \omega_{\hat{k}+np}(0) \right|^2, \quad \forall t \in R,$$

where

$$\sigma = \left[\max_{n \in Z} \{-\rho_n\} \right] \left[\min_{n \in Z} \{-\rho_n\} \right]^{-1}, \quad 0 < \sigma < \infty.$$

Proof: By lemma III.1, $I_{\hat{k}}$ is a constant of motion for the invariant subsystem (II.3); then

$$\sum_{n \in Z} \rho_n \left| \omega_{\hat{k}+np}(t) \right|^2 = \sum_{n \in Z} \rho_n \left| \omega_{\hat{k}+np}(0) \right|^2, \quad \forall t \in R. \quad (\text{III.2})$$

If $\Sigma_{\hat{k}} \cap \bar{D}_{|p|} = \emptyset$, then

$$|\hat{k} + np| > |p|, \quad \forall n \in Z.$$

Thus, there exists a constant $\delta > 0$, such that

$$\delta < -\rho_n < 2. \quad (\text{III.3})$$

By (III.2),

$$\begin{aligned} \min_{n \in Z} \{-\rho_n\} \sum_{n \in Z} \left| \omega_{\hat{k}+np}(t) \right|^2 &\leq - \sum_{n \in Z} \rho_n \left| \omega_{\hat{k}+np}(t) \right|^2 \\ &= - \sum_{n \in Z} \rho_n \left| \omega_{\hat{k}+np}(0) \right|^2 \\ &\leq \max_{n \in Z} \{-\rho_n\} \sum_{n \in Z} \left| \omega_{\hat{k}+np}(0) \right|^2, \end{aligned}$$

that is,

$$\sum_{n \in Z} \left| \omega_{\hat{k}+np}(t) \right|^2 \leq \sigma \sum_{n \in Z} \left| \omega_{\hat{k}+np}(0) \right|^2,$$

where

$$\sigma = \left[\max_{n \in Z} \{-\rho_n\} \right] \left[\min_{n \in Z} \{-\rho_n\} \right]^{-1}.$$

By relation (III.3),

$$\frac{1}{2} \delta < \sigma < 2\delta^{-1}.$$

This completes the proof of the theorem. \square

Remark III.1 If $\hat{k} \parallel p$, i.e. \exists real scalar α such that $\hat{k} = \alpha p$, then the invariant subsystem (II.3) reduces to

$$\dot{\omega}_{\hat{k}+np} = 0, \quad \forall n \in Z;$$

thus, it is obviously Liapunov stable for all $t \in R$, in fact, this is a linearization inside the equilibrium space E_p^1 of the 2D Euler equation (cf: Definition 1).

If $|\hat{k}| = |p|$, then the invariant subsystem (II.3) decomposes into two decoupled systems,

$$\begin{aligned} \dot{\omega}_{\hat{k}+np} = & A(p, \hat{k} + (n-1)p) \Gamma \omega_{\hat{k}+(n-1)p} \\ & + A(-p, \hat{k} + (n+1)p) \bar{\Gamma} \omega_{\hat{k}+(n+1)p}, \quad (n \geq 1), \end{aligned} \quad (\text{III.4})$$

$$\begin{aligned} \dot{\omega}_{\hat{k}+np} = & A(p, \hat{k} + (n-1)p) \Gamma \omega_{\hat{k}+(n-1)p} \\ & + A(-p, \hat{k} + (n+1)p) \bar{\Gamma} \omega_{\hat{k}+(n+1)p}, \quad (n \leq -1), \end{aligned} \quad (\text{III.5})$$

where $A(p, \hat{k}) = 0$. The equation for $\omega_{\hat{k}}$ is

$$\dot{\omega}_{\hat{k}} = A(p, \hat{k} - p) \Gamma \omega_{\hat{k}-p} + A(-p, \hat{k} + p) \bar{\Gamma} \omega_{\hat{k}+p}. \quad (\text{III.6})$$

Each of (III.4) and (III.5) is a Hamiltonian system with the Hamiltonian,

$$\begin{aligned} \mathcal{H}_{\hat{k}}^+ &= - \left| \begin{array}{cc} p_1 & \hat{k}_1 \\ p_2 & \hat{k}_2 \end{array} \right| \text{Im} \left\{ \sum_{n=1}^{\infty} \Gamma \rho_n \rho_{n-1} \omega_{\hat{k}+(n-1)p} \bar{\omega}_{\hat{k}+np} \right\}, \\ \mathcal{H}_{\hat{k}}^- &= - \left| \begin{array}{cc} p_1 & \hat{k}_1 \\ p_2 & \hat{k}_2 \end{array} \right| \text{Im} \left\{ \sum_{n=-1}^{-\infty} \Gamma \rho_n \rho_{n-1} \omega_{\hat{k}+(n-1)p} \bar{\omega}_{\hat{k}+np} \right\}; \end{aligned}$$

which has the same representation as (II.5). Denote by $I_{\hat{k}}^+$ and $I_{\hat{k}}^-$ the restrictions of $I_{\hat{k}}$ to the systems (III.4) and (III.5):

$$I_{\hat{k}}^+ = \sum_{n=1}^{\infty} \rho_n \left| \omega_{\hat{k}+np} \right|^2, \quad (III.7)$$

$$I_{\hat{k}}^- = \sum_{n=-1}^{-\infty} \rho_n \left| \omega_{\hat{k}+np} \right|^2. \quad (III.8)$$

Lemma III.2 *If $|\hat{k}| = |p|$, then $I_{\hat{k}}^+$ and $I_{\hat{k}}^-$ are respectively constants of motion for the systems (III.4) and (III.5).*

Proof: The same with that for Lemma III.1 . \square

Theorem III.2 (Half Class Stability Theorem) *If $|\hat{k}| = |p|$, $\hat{k} + p \notin \bar{D}_{|p|}$, then the linear Hamiltonian system (III.4) is Liapunov stable for all $t \in R$, in fact,*

$$\sum_{n=1}^{\infty} \left| \omega_{\hat{k}+np}(t) \right|^2 \leq \sigma \sum_{n=1}^{\infty} \left| \omega_{\hat{k}+np}(0) \right|^2, \quad \forall t \in R,$$

where

$$\sigma = \left[\max_{n \geq 1} \{-\rho_n\} \right] \left[\min_{n \geq 1} \{-\rho_n\} \right]^{-1}, \quad 0 < \sigma < \infty.$$

If $|\hat{k}| = |p|$, $\hat{k} - p \notin \bar{D}_{|p|}$, then the linear Hamiltonian system (III.5) is Liapunov stable for all $t \in R$, in fact,

$$\sum_{n=-1}^{-\infty} \left| \omega_{\hat{k}+np}(t) \right|^2 \leq \sigma \sum_{n=-1}^{-\infty} \left| \omega_{\hat{k}+np}(0) \right|^2, \quad \forall t \in R,$$

where

$$\sigma = \left[\max_{n \leq -1} \{-\rho_n\} \right] \left[\min_{n \leq -1} \{-\rho_n\} \right]^{-1}, \quad 0 < \sigma < \infty.$$

Proof: The same argument on $I_{\hat{k}}^+$ and $I_{\hat{k}}^-$ as that on $I_{\hat{k}}$ in the proof of Theorem III.1. \square

Remark III.2 *If $\hat{k} = (p_2, -p_1)^T$ or $\hat{k} = (-p_2, p_1)^T$, then $\hat{k} + p \notin \bar{D}_{|p|}$ and $\hat{k} - p \notin \bar{D}_{|p|}$. Therefore, by Theorem III.2, both systems (III.4) and (III.5) are Liapunov stable for all $t \in R$. Points in $\Sigma_{\hat{k}}$ are on the tangent lines to the circle of radius $|p|$ at \hat{k} (cf: Fig.2).*

IV. Properties of the Point Spectrum for the Linearized Two-Dimensional Euler Equation as a Linear Hamiltonian System

In this section, we study the properties of the eigenvalues for the linear Hamiltonian system

(II.6). The right hand side of (II.6) defines a linear operator denoted by \mathcal{L} , i.e.

$$\mathcal{L} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \quad (\text{IV.1})$$

where

$$\begin{aligned} \alpha &= (\cdots \alpha_{-1} \ \alpha_0 \ \alpha_1 \cdots)^T, & \beta &= (\cdots \beta_{-1} \ \beta_0 \ \beta_1 \cdots)^T; \\ \xi &= (\cdots \xi_{-1} \ \xi_0 \ \xi_1 \cdots)^T, & \eta &= (\cdots \eta_{-1} \ \eta_0 \ \eta_1 \cdots)^T; \\ \xi_n &= \frac{1}{2} \rho_n^{-1} \frac{\partial \mathcal{H}_{\hat{k}}}{\partial \beta_n}, & \eta_n &= -\frac{1}{2} \rho_n^{-1} \frac{\partial \mathcal{H}_{\hat{k}}}{\partial \alpha_n}, \quad n \in Z. \end{aligned}$$

\mathcal{L} has the infinite matrix representation:

$$\kappa \begin{pmatrix} \begin{array}{c|c} n \rightarrow -\infty & n \rightarrow -\infty \\ \hline \begin{array}{cccc} \ddots & \ddots & \ddots & \\ & -u_{n-1} & 0 & u_{n+1} \\ & \bigcirc & & \bigcirc \end{array} & \begin{array}{cccc} \ddots & \ddots & \ddots & \\ & v_{n-1} & 0 & v_{n+1} \\ & \bigcirc & & \bigcirc \end{array} \\ \hline n \rightarrow +\infty & n \rightarrow +\infty \end{array} \\ \hline \begin{array}{c|c} n \rightarrow -\infty & n \rightarrow -\infty \\ \hline \begin{array}{cccc} \ddots & \ddots & \ddots & \\ & -v_{n-1} & 0 & -v_{n+1} \\ & \bigcirc & & \bigcirc \end{array} & \begin{array}{cccc} \ddots & \ddots & \ddots & \\ & -u_{n-1} & 0 & u_{n+1} \\ & \bigcirc & & \bigcirc \end{array} \\ \hline n \rightarrow +\infty & n \rightarrow +\infty \end{array} \end{pmatrix}$$

where $u_n = \Gamma_r \rho_n$, $v_n = \Gamma_i \rho_n$, $\kappa = \frac{1}{2} \begin{vmatrix} \hat{k}_1 & p_1 \\ \hat{k}_2 & p_2 \end{vmatrix}$. We define the enstrophy norm which is the ℓ_2 norm:

$$\left\| (\alpha, \beta)^T \right\|^2 = \sum_{n \in Z} (\alpha_n^2 + \beta_n^2). \quad (\text{IV.2})$$

Lemma IV.1 *The linear operator \mathcal{L} maps $\ell_2 \times \ell_2$ into $\ell_2 \times \ell_2$:*

$$\mathcal{L} : \ell_2 \times \ell_2 \mapsto \ell_2 \times \ell_2 .$$

Proof: Notice that

$$\rho_n \longrightarrow |p|^{-2} , \quad \text{as } |n| \longrightarrow \infty .$$

Let

$$\rho_* = \max_{n \in \mathbb{Z}} |\rho_n|$$

be the maximum of $|\rho_n|$; then $\rho_* < \infty$, and from (IV.1), we have

$$|\xi_n| \leq c \left\{ |\alpha_{n+1}| + |\alpha_{n-1}| + |\beta_{n+1}| + |\beta_{n-1}| \right\} ,$$

$$|\eta_n| \leq c \left\{ |\alpha_{n+1}| + |\alpha_{n-1}| + |\beta_{n+1}| + |\beta_{n-1}| \right\} ,$$

where $c = \frac{1}{2}\rho_* |\Gamma| \left| \hat{k}_1 p_2 - \hat{k}_2 p_1 \right|$. Thus

$$\left\| \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\|^2 \leq 8c^2 \left\| \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\|^2 ,$$

which proves the lemma. \square

A complex number λ is an eigenvalue of \mathcal{L} , if there exists $(\alpha, \beta)^T \in \ell_2 \times \ell_2$ such that

$$\mathcal{L} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix} . \tag{IV.3}$$

Lemma IV.2 *The eigenvalues of the linear operator \mathcal{L} have the following properties:*

1. *If $\lambda \in \mathbb{C}$ is an eigenvalue of \mathcal{L} , then both $\bar{\lambda}$ (the complex conjugate of λ) and $-\lambda$ are also eigenvalues.*

2. If λ is a real eigenvalue, then λ is a multiple eigenvalue.

3. If λ is a simple eigenvalue which is not real, then its corresponding eigenvector satisfies the relation $\beta = \pm i\alpha$.

Proof: Since \mathcal{L} is a real linear operator, if λ is an eigenvalue of \mathcal{L} , then $\bar{\lambda}$ is also an eigenvalue. Next we show that if λ is an eigenvalue of \mathcal{L} , then $-\lambda$ is also an eigenvalue. Let λ and ω be an eigenvalue and a corresponding eigenvector, starting from the equation (II.3), we have

$$\begin{aligned} \lambda \omega_{\hat{k}+np} &= A(p, \hat{k} + (n-1)p) \Gamma \omega_{\hat{k}+(n-1)p} \\ &\quad + A(-p, \hat{k} + (n+1)p) \bar{\Gamma} \omega_{\hat{k}+(n+1)p} . \end{aligned}$$

Then,

$$\hat{\omega}_{\hat{k}+np} = (-1)^n \omega_{\hat{k}+np}$$

satisfies

$$\begin{aligned} (-\lambda) \hat{\omega}_{\hat{k}+np} &= A(p, \hat{k} + (n-1)p) \Gamma \hat{\omega}_{\hat{k}+(n-1)p} \\ &\quad + A(-p, \hat{k} + (n+1)p) \bar{\Gamma} \hat{\omega}_{\hat{k}+(n+1)p} . \end{aligned}$$

Therefore, $-\lambda$ is also an eigenvalue. To prove claims 2 and 3, notice that the Hamiltonian $\mathcal{H}_{\hat{k}}$ (II.6) is invariant under the transformation

$$\begin{cases} \tilde{\alpha}_n &= -\beta_n, \\ \tilde{\beta}_n &= \alpha_n. \end{cases}$$

Therefore, if λ is an eigenvalue and

$$e^{\lambda t} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

solves the linear system (II.6), then

$$e^{\lambda t} \begin{pmatrix} -\beta \\ \alpha \end{pmatrix}$$

also solves the system (II.6). If λ is real, then $(\alpha, \beta)^T$ can be chosen to be real. Assume that

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \eta \begin{pmatrix} -\beta \\ \alpha \end{pmatrix}, \tag{IV.4}$$

then $\eta^2 = -1$ which leads to a contradiction. Thus $(\alpha, \beta)^T$ and $(-\beta, \alpha)^T$ are linearly independent, and λ is a multiple eigenvalue which proves claim 2. If λ is simple and not real, then $(\alpha, \beta)^T$ and $(-\beta, \alpha)^T$ are linearly dependent and (IV.4) implies that $\beta = \pm i\alpha$, which proves claim 3. \square

In fact, we have the following theorem.

Theorem IV.1 *The eigenvalues of the linear operator \mathcal{L} defined in (IV.1), have the following properties:*

1. *The eigenvalues of \mathcal{L} are of four types: real pairs $(c, -c)$, purely imaginary pairs $(id, -id)$, quadruples $(\pm c \pm id)$, and zero eigenvalues.*
2. *If λ is a real eigenvalue, then λ is a multiple eigenvalue. If λ is a simple non-real eigenvalue, then its corresponding eigenvector satisfies the relation $\beta = \pm i\alpha$.*
3. *If $\Sigma_{\hat{k}} \cap \bar{D}_{|p|} = \emptyset$, then all the eigenvalues of \mathcal{L} are either purely imaginary and in complex conjugate pairs ($\lambda = ic, -ic$; c is real and not zero) or zeros.*
4. *If $|\hat{k}| = |p|$, then zero is a multiple eigenvalue of \mathcal{L} .*
5. *If $|\hat{k}| = |p|$ and $\hat{k} + p \notin \bar{D}_{|p|}$ (or $\hat{k} - p \notin \bar{D}_{|p|}$), then all the eigenvalues for the system (III.4) (or (III.5)) are either purely imaginary and in complex conjugate pairs or zeros.*

Proof: Claims 1 and 2 are proved in Lemma IV.2. Next we prove claim 3. If $\Sigma_{\hat{k}} \cap \bar{D}_{|p|} = \emptyset$, then $I_{\hat{k}}$ is negative definite for $(\alpha, \beta)^T \in \ell_2 \times \ell_2$. In fact, there exists a constant $\delta > 0$, such that

$$\delta < -\rho_n < 2, \quad \forall n \in Z. \quad (\text{IV.5})$$

Thus,

$$-I_{\hat{k}} > \delta \|(\alpha, \beta)^T\|^2. \quad (\text{IV.6})$$

Let λ be an eigenvalue of \mathcal{L} , and $(\tilde{\alpha}, \tilde{\beta})^T$ be its corresponding eigenvector, which are written in terms of real and imaginary parts,

$$\lambda = \lambda_r + i \lambda_i, \quad (\tilde{\alpha}, \tilde{\beta})^T = (\tilde{\alpha}^{(1)}, \tilde{\beta}^{(1)})^T + i (\tilde{\alpha}^{(2)}, \tilde{\beta}^{(2)})^T. \quad (\text{IV.7})$$

Then both the real and the imaginary parts of $e^{\lambda t}(\tilde{\alpha}, \tilde{\beta})^T$, denoted by $(\alpha^{(1)}, \beta^{(1)})$ and $(\alpha^{(2)}, \beta^{(2)})$, are real solutions to system (II.6), where

$$\alpha^{(1)} = e^{\lambda_r t} [\tilde{\alpha}^{(1)} \cos \lambda_i t - \tilde{\alpha}^{(2)} \sin \lambda_i t],$$

$$\beta^{(1)} = e^{\lambda_r t} [\tilde{\beta}^{(1)} \cos \lambda_i t - \tilde{\beta}^{(2)} \sin \lambda_i t];$$

$$\alpha^{(2)} = e^{\lambda_r t} [\tilde{\alpha}^{(1)} \sin \lambda_i t + \tilde{\alpha}^{(2)} \cos \lambda_i t],$$

$$\beta^{(2)} = e^{\lambda_r t} [\tilde{\beta}^{(1)} \sin \lambda_i t + \tilde{\beta}^{(2)} \cos \lambda_i t].$$

Denote by $I_{\hat{k}}^{(j)}$ ($j = 1, 2$) the values of $I_{\hat{k}}$ evaluated at $(\alpha^{(j)}, \beta^{(j)})^T$; then

$$I_{\hat{k}}^{(1)}(t) + I_{\hat{k}}^{(2)}(t) = e^{2\lambda_r t} \sum_{n \in Z} \rho_n [(\tilde{\alpha}^{(1)})^2 + (\tilde{\alpha}^{(2)})^2 + (\tilde{\beta}^{(1)})^2 + (\tilde{\beta}^{(2)})^2]$$

$$\begin{aligned}
&= I_{\hat{k}}^{(1)}(0) + I_{\hat{k}}^{(2)}(0) \\
&= \sum_{n \in Z} \rho_n \left[(\tilde{\alpha}^{(1)})^2 + (\tilde{\alpha}^{(2)})^2 + (\tilde{\beta}^{(1)})^2 + (\tilde{\beta}^{(2)})^2 \right] \\
&\neq 0,
\end{aligned}$$

by Lemma III.1 and relations (IV.5, IV.6). Thus,

$$e^{2\lambda_r t} = 1, \quad \text{i.e. } \lambda_r = 0.$$

This proves claim 3. To prove claim 4, notice that if $|\hat{k}| = |p|$, then system (II.3) decomposes into systems (III.4, III.5, III.6). Thus the vectors $(\alpha, \beta)^T$ defined as

$$\alpha_0 = 1, \quad \alpha_n = 0 \quad (n \neq 0), \quad \beta_n = 0 \quad (\forall n \in Z);$$

and

$$\beta_0 = 1, \quad \beta_n = 0 \quad (n \neq 0), \quad \alpha_n = 0 \quad (\forall n \in Z);$$

give two linearly independent eigenvectors of \mathcal{L} with eigenvalue zero. This proves claim 4.

Claim 5 follows from the proof for claim 3 when restricted to system (III.4) or (III.5). \square

Remark IV.1 *For a finite dimensional linear Hamiltonian system, it is well-known that the eigenvalues are of four types: real pairs $(c, -c)$, purely imaginary pairs $(id, -id)$, quadruples $(\pm c \pm id)$, and zero eigenvalues [12] [13] [14]. There is also a complete theorem on the normal forms of such Hamiltonians [14].*

V. The Point Spectrum of the Linearized Two-Dimensional Euler Equation: A Continued Fraction Study

Rewrite the equation (II.3) as follows,

$$\rho_n^{-1} \dot{\tilde{z}}_n = a \left[\tilde{z}_{n+1} - \tilde{z}_{n-1} \right] , \quad (\text{V.1})$$

where $\tilde{z}_n = \rho_n e^{in(\theta+\pi/2)} \omega_{\hat{k}+np}$, $\theta + \gamma = \pi/2$, $\Gamma = |\Gamma| e^{i\gamma}$, $a = \frac{1}{2} |\Gamma| \left| \begin{smallmatrix} p_1 & \hat{k}_1 \\ p_2 & \hat{k}_2 \end{smallmatrix} \right|$, $\rho_n = |\hat{k} + np|^{-2} - |p|^{-2}$. Let $\tilde{z}_n = e^{\lambda t} z_n$, where $\lambda \in C$; then z_n satisfies

$$a_n z_n + z_{n-1} - z_{n+1} = 0 , \quad (\text{V.2})$$

where $a_n = \lambda(a\rho_n)^{-1}$. Let $w_n = z_n/z_{n-1}$ [15]; then w_n satisfies

$$a_n + \frac{1}{w_n} = w_{n+1} . \quad (\text{V.3})$$

Iteration of (V.3) leads to the continued fraction solution [15],

$$w_n^{(1)} = a_{n-1} + \frac{1}{a_{n-2} + \frac{1}{a_{n-3} + \ddots}} . \quad (\text{V.4})$$

Rewrite (V.3) as follows,

$$w_n = \frac{1}{-a_n + w_{n+1}} . \quad (\text{V.5})$$

Iteration of (V.5) leads to the continued fraction solution [15],

$$w_n^{(2)} = -\frac{1}{a_n + \frac{1}{a_{n+1} + \frac{1}{a_{n+2} + \ddots}}} . \quad (\text{V.6})$$

Before we study the two continued fraction solutions (V.4) and (V.6), we like to quote two theorems on the convergence of continued fractions [21].

Theorem V.1 (Śleszyński-Pringsheim's Theorem) *The continued fraction*

$$\mathbf{K}(a_n/b_n) = \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \ddots}}} ,$$

where $\{a_n\}$ and $\{b_n\}$ are complex numbers and all $a_n \neq 0$, converges if for all n

$$|b_n| \geq |a_n| + 1 .$$

Under the same condition

$$|\mathbf{K}(a_n/b_n)| \leq 1 .$$

Theorem V.2 (Van Vleck's Theorem) Let $0 < \epsilon < \pi/2$, and let b_n satisfy

$$-\pi/2 + \epsilon < \arg\{b_n\} < \pi/2 - \epsilon$$

for all n . Then the continued fraction $\mathbf{K}(1/b_n)$ converges if and only if

$$\sum_{n=1}^{\infty} |b_n| = \infty .$$

Notice that as $n \rightarrow \pm\infty$,

$$a_n \rightarrow \tilde{a} = -\lambda a^{-1}|p|^2 . \quad (\text{V.7})$$

Then we have the corollary.

Corollary 1 If $\text{Re}\{\tilde{a}\} \neq 0$, or $\text{Re}\{\tilde{a}\} = 0$ ($|\tilde{a}| > 2$), then the two continued fractions (V.4) and (V.6) converge.

Proof: If $\text{Re}\{\tilde{a}\} \neq 0$, then there exists an positive integer \tilde{N} and a positive constant ϵ , such that

$$-\pi/2 + \epsilon < \arg\{a_n\} < \pi/2 - \epsilon$$

for all $|n| \geq \tilde{N}$, or

$$-\pi/2 + \epsilon < \arg\{-a_n\} < \pi/2 - \epsilon$$

for all $|n| \geq \tilde{N}$. In either case, applying Van Vleck's theorem, we have the convergence of the two continued fractions (V.4) and (V.6). If $\text{Re}\{\tilde{a}\} = 0$ ($|\tilde{a}| > 2$), then there exists an positive integer \hat{N} such that

$$|a_n| > 2$$

for all $|n| \geq \hat{N}$. Then applying Śleszyński-Pringsheim's theorem, we have the convergence of the two continued fractions (V.4) and (V.6). \square

Remark V.1 *In fact, as proved in Theorem VI.5, $\text{Re}\{\tilde{a}\} = 0$ and $|\tilde{a}| \leq 2$ correspond to the continuous spectrum (= essential spectrum) of the system.*

When $\text{Re}\{\tilde{a}\} \neq 0$, or $\text{Re}\{\tilde{a}\} = 0$ ($|\tilde{a}| > 2$), as $n \rightarrow -\infty$,

$$w_n^{(1)} \rightarrow w^{(1)} = \tilde{a} + \frac{1}{\tilde{a} + \frac{1}{\tilde{a} + \frac{1}{\ddots}}} = \tilde{a} + \mathbf{K}(1/\tilde{a}) , \quad (\text{V.8})$$

as $n \rightarrow +\infty$,

$$w_n^{(2)} \rightarrow w^{(2)} = -\frac{1}{\tilde{a} + \frac{1}{\tilde{a} + \frac{1}{\ddots}}} = -\mathbf{K}(1/\tilde{a}) . \quad (\text{V.9})$$

Both $w^{(1)}$ and $w^{(2)}$ satisfy the equation,

$$\tilde{w}^2 - \tilde{a}\tilde{w} - 1 = 0 . \quad (\text{V.10})$$

- When $\text{Re}\{\tilde{a}\} \neq 0$, the solutions of (V.10) can be written as:

$$w_{\pm} = \frac{1}{2} \left[\tilde{a} \pm \delta \sqrt{\tilde{a}^2 + 4} \right] , \quad (\text{V.11})$$

where $\delta = \text{sign}(\text{Re}\{\tilde{a}\}) \text{sign}(\text{Re}\{\sqrt{\tilde{a}^2 + 4}\})$. (Note that if $\text{Re}\{\tilde{a}\} \neq 0$, then $\text{Re}\{\sqrt{\tilde{a}^2 + 4}\} \neq 0$.) ■

- When $\text{Re}\{\tilde{a}\} = 0$ ($|\tilde{a}| > 2$), let $\tilde{a} = i\xi$, ξ is a real number, the solutions of (V.10) can be written as:

$$w_{\pm} = \frac{i}{2} \left[\xi \pm \delta \sqrt{\xi^2 - 4} \right] , \quad (\text{V.12})$$

where $\delta = \text{sign}(\xi) \text{sign}(\sqrt{\xi^2 - 4})$.

Lemma V.1 *The solutions (V.11) and (V.12) satisfy the inequality*

$$|w_+| > 1 > |w_-|,$$

and the continued fractions (V.8) and (V.9) have the values

$$w^{(1)} = w_+ , \quad w^{(2)} = w_- .$$

Proof: First we show that $|w_+| > 1$ when $\text{Re}\{\tilde{a}\} \neq 0$.

$$w_+ \overline{w_+} = \frac{1}{4} \left(|\tilde{a}|^2 + |\tilde{a}^2 + 4| + \tilde{a} \delta \sqrt{\tilde{a}^2 + 4} + \tilde{a} \delta \sqrt{\tilde{a}^2 + 4} \right) . \quad (\text{V.13})$$

Let $\tilde{a} = a_1 + ia_2$, $\sqrt{\tilde{a}^2 + 4} = b_1 + ib_2$; then

$$b_1 b_2 = a_1 a_2 , \quad (\text{V.14})$$

$$b_1^2 - b_2^2 = a_1^2 - a_2^2 + 4 . \quad (\text{V.15})$$

From (V.14), we have

$$a_1^2(a_2 b_2) = b_2^2(a_1 b_1) .$$

Thus, $a_2 b_2$ is either zero or of the same sign as $a_1 b_1$. Therefore,

$$\tilde{a} \delta \sqrt{\tilde{a}^2 + 4} + \tilde{a} \delta \sqrt{\tilde{a}^2 + 4} = 2\delta[a_1 b_1 + a_2 b_2] > 0 . \quad (\text{V.16})$$

Together with

$$|\tilde{a}^2 + 4| \geq 4 - |\tilde{a}|^2 ,$$

we have $|w_+| > 1$. When $\operatorname{Re}\{\tilde{a}\} = 0$ ($|\tilde{a}| > 2$), it is obvious that $|w_+| > 1$. Notice that $w_+w_- = -1$, we have $|w_-| < 1$ in both cases. Thus, we have

$$|w_+| > 1 > |w_-| .$$

From the relation $w_+w_- = -1$, when $\operatorname{Re}\{\tilde{a}\} \neq 0$, $\operatorname{Re}\{w_+\}$ and $\operatorname{Re}\{w_-\}$ are of opposite signs. When $\operatorname{Re}\{\tilde{a}\} \neq 0$, $\operatorname{Re}\{w^{(1)}\}$ and $\operatorname{Re}\{w^{(2)}\}$ are of opposite signs, and $\operatorname{Re}\{w^{(1)}\}$ and $\operatorname{Re}\{w_+\}$ are of the same sign; thus, $w^{(1)} = w_+$ and $w^{(2)} = w_-$. When $\operatorname{Re}\{\tilde{a}\} = 0$ ($|\tilde{a}| > 2$), by Śleszyński-Pringsheim's theorem,

$$|\mathbf{K}(1/\tilde{a})| \leq 1 ;$$

then,

$$|w^{(1)}| = |\tilde{a} + \mathbf{K}(1/\tilde{a})| \geq |\tilde{a}| - |\mathbf{K}(1/\tilde{a})| > 1 .$$

Thus, $w^{(1)} = w_+$ and $w^{(2)} = w_-$. \square

Definition 5 Define $w^{(*)}$ as follows

$$w_n^{(*)} = w_n^{(1)} \text{ for } n \leq 1, \quad w_n^{(*)} = w_n^{(2)} \text{ for } n > 1.$$

Then $w_n^{(*)}$ solves (V.3), provided that $w_1^{(1)} = w_1^{(2)}$, i.e.

$$f = a_0 + \left(\frac{1}{a_{-1} + \frac{1}{a_{-2} + \frac{1}{a_{-3} + \ddots}}} \right) + \left(\frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ddots}}} \right) = 0 , \quad (\text{V.17})$$

where $f = f(\tilde{\lambda}, \hat{k}, p)$, $\tilde{\lambda} = \lambda/a$. Let $z^{(*)}$ satisfy

$$w_n^{(*)} = z_n^{(*)} / z_{n-1}^{(*)};$$

then as $n \rightarrow +\infty$,

$$z_n^{(*)} \sim (w_-)^n ,$$

and as $n \rightarrow -\infty$,

$$z_n^{(*)} \sim (w_+)^n .$$

Thus by Lemma V.1, $z^{(*)} \in \ell_2$. Therefore equation (V.17) determines eigenvalues.

If $\Sigma_{\hat{k}} \cap \bar{D}_{|p|} = \emptyset$, then $\rho_n < 0$ for any $n \in Z$. If $\text{Re}\{\tilde{\lambda}\} \neq 0$, then $\text{Re}\{\tilde{a}_n\} \neq 0$ and are of a fixed sign for any $n \in Z$. Then $\text{Re}\{f\} \neq 0$. Therefore, in such cases, there is no eigenvalue with nonzero real part. This fact is already obtained in Theorem IV.1. Moreover, we have the following fact.

Lemma V.2 *If $\Sigma_{\hat{k}} \cap \bar{D}_{|p|} = \emptyset$, then equation (V.17) determines no eigenvalue.*

Proof: As discussed above, if $\Sigma_{\hat{k}} \cap \bar{D}_{|p|} = \emptyset$, then the possible solution $\tilde{\lambda}$ to (V.17) has to be imaginary. Therefore, \tilde{a} has to be imaginary. Rewrite equation (V.2) as follows

$$\tilde{L}z_n \equiv \frac{\rho_n}{\rho}[z_{n+1} - z_{n-1}] = \tilde{a}z_n . \quad (\text{V.18})$$

If $\Sigma_{\hat{k}} \cap \bar{D}_{|p|} = \emptyset$, then $0 < \rho_n/\rho < 1$ for all $n \in Z$. Thus, $\|\tilde{L}\| \leq 2$. Then if $\tilde{\lambda}$ is an eigenvalue, $|\tilde{a}| \leq \|\tilde{L}\| \leq 2$. Notice that equation (V.17) should be solved under the condition $\text{Re}\{\tilde{a}\} \neq 0$ or $\text{Re}\{\tilde{a}\} = 0$ ($|\tilde{a}| > 2$); thus, in this case, equation (V.17) determines no eigenvalue. \square

Remark V.2 *In fact, as proved in Theorem VI.5, $\text{Re}\{\tilde{a}\} = 0$ and $|\tilde{a}| \leq 2$ correspond to the continuous spectrum (= essential spectrum) of the system. Thus, if $\Sigma_{\hat{k}} \cap \bar{D}_{|p|} = \emptyset$, the point spectrum is empty (cf: Theorem VI.5). Then the problem is reduced to solving equation (V.17) under the conditions $\Sigma_{\hat{k}} \cap \bar{D}_{|p|} \neq \emptyset$, $\text{Re}\{\tilde{a}\} \neq 0$ or $\text{Re}\{\tilde{a}\} = 0$ ($|\tilde{a}| > 2$).*

Example: Let $p = (1, 1)^T$, in this case, only one class $\Sigma_{\hat{k}}$ labeled by $\hat{k} = (1, 0)^T$ has no empty intersection with $\bar{D}_{|p|}$ (the other class labeled by $\hat{k} = (0, 1)^T$ gives the complex

conjugate of the system led by the class labeled by $\hat{k} = (1, 0)^T$. For this class, $|\rho_n/\rho| \leq 1$ for all $n \in Z$. Thus, the linear operator \tilde{L} defined in (V.18) has norm $\|\tilde{L}\| \leq 2$. Therefore, equation (V.17) determines no real eigenvalue. Numerical calculation on equation (V.17) gives the eigenvalue:

$$\tilde{\lambda} = 0.24822302478255 + i 0.35172076526520 .$$

By Theorem IV.1, equation (V.17) determines a quadruple of eigenvalues, see figure 3 for an illustration.

VI. The Spectra of the Linearized Two-Dimensional Euler Equation: An Infinite Matrix Study

VI.1. The General Setup

Rewrite (II.3) as follows:

$$\dot{\tilde{z}}_n = ia \left[\rho_{n-1} \tilde{z}_{n-1} + \rho_{n+1} \tilde{z}_{n+1} \right], \quad (n \in Z) \quad (\text{VI.1})$$

where

$$\tilde{z}_n = e^{in\theta} \omega_{\hat{k}+np}, \quad \Gamma = |\Gamma| e^{i\gamma}, \quad \theta + \gamma = \pi/2, \quad a = \frac{1}{2} |\Gamma| \begin{vmatrix} p_1 & \hat{k}_1 \\ p_2 & \hat{k}_2 \end{vmatrix}.$$

Relabel $\{\tilde{z}_n\}$ as follows:

$$\begin{cases} \tilde{z}_n = z_{2n}, & n \geq 1, \\ \tilde{z}_{-n} = z_{2n+1}, & n \geq 0; \end{cases}$$

then

$$\dot{z}_{2n} = ia \left[\rho_{n-1} z_{2(n-1)} + \rho_{n+1} z_{2(n+1)} \right], \quad (n \geq 2) \quad (\text{VI.2})$$

$$\dot{z}_{2n+1} = ia \left[\rho_{-n+1} z_{2(n-1)+1} + \rho_{-n-1} z_{2(n+1)+1} \right], \quad (n \geq 1) \quad (\text{VI.3})$$

$$\dot{z}_2 = ia \left[\rho_0 z_1 + \rho_2 z_4 \right], \quad (\text{VI.4})$$

$$\dot{z}_1 = ia \left[\rho_1 z_2 + \rho_{-1} z_3 \right], \quad (\text{VI.5})$$

for $z = (z_1, z_2, \dots)^T$. Notice that Equations (VI.2) and (VI.3) are decoupled, the coupling between components of z with even and odd indices is through Equations (VI.4) and (VI.5). The right hand side of (VI.2)–(VI.5) define a bounded linear operator $\mathcal{L}_A : \ell_2 \mapsto \ell_2$, with the infinite matrix representation,

$$A = ia \begin{pmatrix} 0 & \rho_1 & \rho_{-1} & 0 & 0 & 0 & 0 & \\ \rho_0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & \\ \rho_0 & 0 & 0 & 0 & \rho_{-2} & 0 & 0 & \bigcirc \\ 0 & \rho_1 & 0 & 0 & 0 & \rho_3 & 0 & \\ 0 & 0 & \rho_{-1} & 0 & 0 & 0 & \rho_{-3} & \\ & \bigcirc & & \ddots & & \bigcirc & & \ddots \end{pmatrix}. \quad (\text{VI.6})$$

More importantly,

$$\rho_n \longrightarrow \rho = -|p|^{-2}, \quad \text{as } |n| \rightarrow \infty. \quad (\text{VI.7})$$

Define the infinite matrix

$$B = ib \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & \bigcirc \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & \\ & \bigcirc & & \ddots & & \bigcirc & & \ddots \end{pmatrix}, \quad (\text{VI.8})$$

where $b = a\rho = -a|p|^{-2}$. Define the infinite matrix C as

$$C = A - B, \quad (\text{VI.9})$$

that is,

$$C = ia \begin{pmatrix} 0 & \tilde{\rho}_1 & \tilde{\rho}_{-1} & 0 & 0 & 0 & 0 & \\ \tilde{\rho}_0 & 0 & 0 & \tilde{\rho}_2 & 0 & 0 & 0 & \\ \tilde{\rho}_0 & 0 & 0 & 0 & \tilde{\rho}_{-2} & 0 & 0 & \bigcirc \\ 0 & \tilde{\rho}_1 & 0 & 0 & 0 & \tilde{\rho}_3 & 0 & \\ 0 & 0 & \tilde{\rho}_{-1} & 0 & 0 & 0 & \tilde{\rho}_{-3} & \\ & \bigcirc & & \ddots & & \bigcirc & & \ddots \end{pmatrix}. \quad (\text{VI.10})$$

where $\tilde{\rho}_n = \rho_n - \rho$. Denote by \mathcal{L}_B and \mathcal{L}_C the bounded linear operators with the infinite matrix representations by B and C . According to Duren [18], \mathcal{L}_A , \mathcal{L}_B and \mathcal{L}_C are called $(2 \times 2 + 1)$ -operators, since their entries $c_{n,n+m}$ satisfy the condition $c_{n,n+m} = 0$ if $|m| > 2$. \mathcal{L}_B is a $(2 \times 2 + 1)$ -operator with constant coefficients, since its entries $c_{n,n+m}$ is independent of n when $n > 2$; and $i\mathcal{L}_B$ is self-adjoint.

Theorem VI.1 *The bounded linear operator $\mathcal{L}_C : \ell_2 \mapsto \ell_2$ is a compact operator.*

Proof: Denote by $\mathcal{L}_C^{(N)}$ the linear operator represented through the matrix $C_{N \times N}$ obtained from C by replacing its entries $c_{m,n}$ by 0, when $m > N$. Let $\{z^{(j)}\}$ be a bounded sequence in ℓ_2 ; then $\{\mathcal{L}_C^{(1)} z^{(j)}\}$ is a bounded sequence in which each element has only one nonzero component, i.e.

$$(\mathcal{L}_C^{(1)} z^{(j)})_n = 0, \quad \text{when } n > 1.$$

Thus, there exists a subsequence $\{z^{(1_j)}\}$, such that $\{\mathcal{L}_C^{(1)} z^{(1_j)}\}$ converges in ℓ_2 . Similarly, we can get a subsequence of $\{z^{(1_j)}\}$, denoted as $\{z^{(2_j)}\}$, such that $\{\mathcal{L}_C^{(2)} z^{(2_j)}\}$ converges in ℓ_2 , and so on. Therefore, we have a nested list of subsequences:

$$\begin{array}{ccc} z^{(1_1)} & z^{(1_2)} & \dots\dots\dots \\ z^{(2_1)} & z^{(2_2)} & \dots\dots\dots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{array}$$

We choose the subsequence $\{z^{(n_n)}\}$ of $\{z^{(j)}\}$, which is the diagonal of the above list. There exist constants ζ and N_0 , such that

$$\left\| \mathcal{L}_C^{(\hat{n})} z^{(n_n)} - \mathcal{L}_C z^{(n_n)} \right\| \leq \frac{\zeta}{\hat{n}^2}, \quad \text{for all } \hat{n} > N_0 \text{ and all } n. \quad (\text{VI.11})$$

For any $\epsilon > 0$, choose \hat{N} large enough, such that

$$\frac{\zeta}{\hat{N}^2} < \frac{1}{3} \epsilon. \quad (\text{VI.12})$$

Since the subsequence $\{\mathcal{L}_C^{(\hat{N})} z^{(\hat{N}_j)}\}$ converges, there exists \tilde{N} , such that

$$\left\| \mathcal{L}_C^{(\hat{N})} z^{(\hat{N}_{j_1})} - \mathcal{L}_C^{(\hat{N})} z^{(\hat{N}_{j_2})} \right\| < \frac{1}{3} \epsilon, \quad \forall j_1, j_2 > \tilde{N}. \quad (\text{VI.13})$$

Let $N_1 = \max \{\hat{N}, \tilde{N}\}$; then

$$\left\| \mathcal{L}_C^{(\hat{N})} z^{(n_n)} - \mathcal{L}_C^{(\hat{N})} z^{(\tilde{n}_{\tilde{n}})} \right\| < \frac{1}{3} \epsilon \quad \forall n, \tilde{n} > N_1. \quad (\text{VI.14})$$

Thus

$$\begin{aligned} \left\| \mathcal{L}_C z^{(n_n)} - \mathcal{L}_C z^{(\tilde{n}_{\tilde{n}})} \right\| &\leq \left\| \mathcal{L}_C z^{(n_n)} - \mathcal{L}_C^{(\hat{N})} z^{(n_n)} \right\| \\ &\quad + \left\| \mathcal{L}_C^{(\hat{N})} z^{(n_n)} - \mathcal{L}_C^{(\hat{N})} z^{(\tilde{n}_{\tilde{n}})} \right\| \\ &\quad + \left\| \mathcal{L}_C^{(\hat{N})} z^{(\tilde{n}_{\tilde{n}})} - \mathcal{L}_C z^{(\tilde{n}_{\tilde{n}})} \right\| \\ &< \frac{1}{3} \epsilon + \frac{1}{3} \epsilon + \frac{1}{3} \epsilon = \epsilon, \quad \forall n, \tilde{n} > N_1. \end{aligned}$$

Therefore, $\left\{ \mathcal{L}_C z^{(n_n)} \right\}$ is a Cauchy sequence in ℓ_2 ; thus converges. This proves that \mathcal{L}_C is a compact operator. \square

Remark VI.1 *In fact, a theorem of Achieser and Glasmann [22] [18] states that a $(2M+1)$ -operator is compact if and only if its diagonal sequence entries tend to zeros, i.e. $c_{n,n+m} \rightarrow 0$, as $n \rightarrow \infty$ for each fixed m , $|m| \leq M$. Here we give the proof for self-containedness.*

VI.2. The Spectra of the Linear Operator \mathcal{L}_B

Next we will follow a theory of Duren [18] to study the spectra of the constant-coefficient infinite-matrix bounded self-adjoint operator $i\mathcal{L}_B$.

The *characteristic polynomial* for the difference equation

$$(B - \lambda I)z = 0, \quad (\text{VI.15})$$

where I is the identity matrix, is defined as:

$$f_B(w, \lambda) = ib - \lambda w^2 + ib w^4. \quad (\text{VI.16})$$

Define the rescaled characteristic polynomial as follows:

$$\tilde{f}_B(w, \tilde{\lambda}) = 1 - \tilde{\lambda} w^2 + w^4, \quad (\text{VI.17})$$

where $\lambda = ib\tilde{\lambda}$. In fact, $\tilde{f}_B(w, \tilde{\lambda})$ is the characteristic polynomial for the difference equation

$$(\tilde{B} - \tilde{\lambda} I)z = 0, \quad (\text{VI.18})$$

where $\tilde{B} = -ib^{-1}B$. The roots of $\tilde{f}_B(w, \tilde{\lambda})$ are:

$$w_*, \quad -w_*, \quad \frac{1}{w_*}, \quad -\frac{1}{w_*}, \quad (\text{VI.19})$$

where

$$w_* = \left[\frac{\tilde{\lambda} + \sqrt{\tilde{\lambda}^2 - 4}}{2} \right]^{1/2}.$$

Definition 6 *The **spectral curve** of the linear operator $\mathcal{L}_{\tilde{B}}$ (with the infinite matrix representation by \tilde{B}), denoted by $C_{\tilde{B}}$, is defined to be the set of all $\tilde{\lambda} \in C$ for which the characteristic polynomial $\tilde{f}_B(w, \tilde{\lambda})$ has a root of modulus one. The **spectral point-set** of the operator $\mathcal{L}_{\tilde{B}}$, denoted by $P_{\tilde{B}}$, is defined to be the set of all $\tilde{\lambda} \in C$ for which the characteristic polynomial $\tilde{f}_B(w, \tilde{\lambda})$ has a multiple root. Denote by $S_{\tilde{B}}(\tilde{\lambda})$ the number of roots of $\tilde{f}_B(w, \tilde{\lambda})$, of modulus less than 1 (counted with multiplicity).*

Notice that

$$\tilde{\lambda} = w_*^2 + w_*^{-2}. \quad (\text{VI.20})$$

Let w_* be a root of modulus 1 (then all the four roots are of modulus 1), $w_* = e^{i\theta}$, $\theta \in [0, 2\pi)$; thus the spectral curve $C_{\tilde{B}}$ is the segment of the real axis,

$$C_{\tilde{B}} : \tilde{\lambda} = 2 \cos 2\theta, \quad \theta \in [0, 2\pi). \quad (\text{VI.21})$$

See Fig.4. The spectral point-set $P_{\tilde{B}}$ consists of two points,

$$P_{\tilde{B}} : \tilde{\lambda} = \pm 2, \quad (\text{VI.22})$$

which are the boundary points of the spectral curve $C_{\tilde{B}}$. At $\tilde{\lambda} = \pm 2$, the four roots of $\tilde{f}_B(w, \tilde{\lambda})$ are,

$$\begin{aligned} \text{at } \tilde{\lambda} &= 2 : & 1, -1, 1, -1; \\ \text{at } \tilde{\lambda} &= -2 : & i, -i, -i, i. \end{aligned}$$

The function $S_{\tilde{B}}(\tilde{\lambda})$ is

$$S_{\tilde{B}}(\tilde{\lambda}) = \begin{cases} 0, & \text{if } \tilde{\lambda} \in C_{\tilde{B}}, \\ 2, & \text{if } \tilde{\lambda} \notin C_{\tilde{B}}. \end{cases} \quad (\text{VI.23})$$

The general solution to (VI.18) is

$$z_n = c_1 w_*^n + c_2 (-w_*)^n + c_3 w_*^{-n} + c_4 (-w_*)^{-n}, \quad (\text{VI.24})$$

$$\text{if } \lambda \notin P_{\tilde{B}},$$

$$z_n = c_1 w_*^n + c_2 n w_*^n + c_3 (-w_*)^n + c_4 n (-w_*)^n, \quad (\text{VI.25})$$

$$\text{if } \lambda \in P_{\tilde{B}} \text{ (then } w_* = 1, i);$$

under the restrictions:

$$-\tilde{\lambda} z_1 + z_2 + z_3 = 0, \quad (\text{VI.26})$$

$$z_1 - \tilde{\lambda} z_2 + z_4 = 0. \quad (\text{VI.27})$$

Lemma VI.1 *The general solution (VI.24, VI.25) is in ℓ_2 if and only if $|w_*| < 1$ and $c_3 = c_4 = 0$ in (VI.24) or $|w_*| > 1$ and $c_1 = c_2 = 0$ in (VI.24).*

Proof: See ([18], pp. 24, Lemma 5). \square

Definition 7 *Let $\mathcal{L} : \ell_2 \mapsto \ell_2$ be a linear operator. The set of points $\sigma_p(\mathcal{L})$ in the complex λ -plane C such that $(\mathcal{L} - \lambda I)$ has no inverse (i.e. $\mathcal{L} - \lambda I$ is not 1-1), is called the point spectrum of \mathcal{L} . The set of points $\sigma_r(\mathcal{L})$ in C such that $(\mathcal{L} - \lambda I)^{-1}$ exists and is a linear operator with domain not everywhere dense is called the residual spectrum of \mathcal{L} . The set of points $\sigma_c(\mathcal{L})$ in C such that $(\mathcal{L} - \lambda I)^{-1}$ exists and is an unbounded linear operator with domain everywhere dense is called the continuous spectrum of \mathcal{L} . The set of points $\rho(\mathcal{L})$ in C such that $(\mathcal{L} - \lambda I)^{-1}$ exists and is a bounded linear operator with domain everywhere dense is called the resolvent set of \mathcal{L} . The set $\sigma(\mathcal{L}) = \sigma_p(\mathcal{L}) \cup \sigma_r(\mathcal{L}) \cup \sigma_c(\mathcal{L})$ is called the spectrum of \mathcal{L} .*

Without loss of generality, assume $|w_*| < 1$. Then $\tilde{\lambda}$ is an eigenvalue if and only if there exists a non-trivial solution (c_1, c_2) to the following system,

$$\begin{pmatrix} -\tilde{\lambda}w_* + w_*^2 + w_*^3 & \tilde{\lambda}w_* + w_*^2 - w_*^3 \\ w_* - \tilde{\lambda}w_*^2 + w_*^4 & -w_* - \tilde{\lambda}w_*^2 + w_*^4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0. \quad (\text{VI.28})$$

Theorem VI.2 *The ℓ_2 point spectrum $\sigma_p(B)$ of the linear operator \mathcal{L}_B is empty.*

Proof: The determinant

$$\begin{aligned} & \det \begin{pmatrix} -\tilde{\lambda}w_* + w_*^2 + w_*^3 & \tilde{\lambda}w_* + w_*^2 - w_*^3 \\ w_* - \tilde{\lambda}w_*^2 + w_*^4 & -w_* - \tilde{\lambda}w_*^2 + w_*^4 \end{pmatrix} \\ &= 2w_*^3 \left[w_*^4 - 2\tilde{\lambda}w_*^2 + (\tilde{\lambda}^2 - 1) \right] = 0, \end{aligned} \tag{VI.29}$$

implies that

$$w_*^4 - 2\tilde{\lambda}w_*^2 + \tilde{\lambda}^2 - 1 = 0, \tag{VI.30}$$

since $w_* \neq 0$. w_* is a root of $\tilde{f}_B(w, \tilde{\lambda})$ (VI.17),

$$w_*^4 - \tilde{\lambda}w_*^2 + 1 = 0. \tag{VI.31}$$

From (VI.30, VI.31), we have

$$w_*^2 = \frac{\tilde{\lambda}^2 - 2}{\tilde{\lambda}}. \tag{VI.32}$$

Notice also that

$$\tilde{\lambda} = w_*^2 + w_*^{-2} = \frac{\tilde{\lambda}^2 - 2}{\tilde{\lambda}} + \frac{\tilde{\lambda}}{\tilde{\lambda}^2 - 2},$$

which implies that $\tilde{\lambda} = \pm 2$. Then $|w_*| = 1$. Thus if $|w_*| < 1$, then Equation (VI.28) has only trivial solution. Therefore, the point spectrum of $\mathcal{L}_{\tilde{B}}$ is empty; equivalently, the point spectrum of \mathcal{L}_B is empty. \square

Theorem VI.3 *The ℓ_2 residual spectrum $\sigma_r(B)$ of the linear operator \mathcal{L}_B is empty.*

Proof: $\tilde{\lambda} \in \sigma_r(\tilde{B})$ if and only if the dimension of the orthocomplement of $(\mathcal{L}_{\tilde{B}} - \tilde{\lambda}I) \circ \ell_2$ is positive and $(\mathcal{L}_{\tilde{B}} - \tilde{\lambda}I)^{-1}$ exists. From the inner product relation

$$\begin{aligned} \langle (\mathcal{L}_{\tilde{B}} - \tilde{\lambda}I)z^{(1)}, z^{(2)} \rangle &= \langle z^{(1)}, (\mathcal{L}_{\tilde{B}}^* - \bar{\tilde{\lambda}}I)z^{(2)} \rangle \\ &= \langle z^{(1)}, (\mathcal{L}_{\tilde{B}} - \bar{\tilde{\lambda}}I)z^{(2)} \rangle, \end{aligned}$$

since $\mathcal{L}_{\tilde{B}}$ is self-adjoint, $\mathcal{L}_{\tilde{B}} = \mathcal{L}_{\tilde{B}}^*$ (the adjoint of $\mathcal{L}_{\tilde{B}}$), where $\langle \cdot, \cdot \rangle$ denotes the inner product over the complex field, we have that if $\tilde{\lambda} \in \sigma_r(\tilde{B})$, then $\bar{\tilde{\lambda}} \in \sigma_p(\tilde{B})$. By Theorem VI.2, $\sigma_p(\tilde{B})$ is empty; thus, $\sigma_r(\tilde{B})$ is empty; equivalently, $\sigma_r(B)$ is empty. \square

Since $i\mathcal{L}_B$ is self-adjoint, this theorem is well-known, but we furnish a short proof here. From (VI.15, VI.18) and (VI.21, VI.22), the spectral curve C_B for the linear operator \mathcal{L}_B is the segment of the imaginary axis,

$$C_B : \lambda = i2b \cos 2\theta, \quad \theta \in [0, 2\pi); \quad (\text{VI.33})$$

the spectral point-set P_B for the linear operator \mathcal{L}_B is

$$P_B : \lambda = \pm i2b, \quad (\text{VI.34})$$

which are the boundary points of the spectral curve C_B .

Theorem VI.4 *The ℓ_2 continuous spectrum $\sigma_c(B)$ of the linear operator \mathcal{L}_B is the spectral curve, $\sigma_c(B) = C_B$. The ℓ_2 resolvent set $\rho(B)$ of the linear operator \mathcal{L}_B is the complement of C_B in the finite complex plane C , $\rho(B) = (C_B)'$.*

Proof: First we show that if $\tilde{\lambda} \in C_{\tilde{B}}$, then $\tilde{\lambda} \in \sigma_c(\tilde{B})$. Since both $\sigma_p(\tilde{B})$ and $\sigma_r(\tilde{B})$ are empty by Theorems VI.2 and VI.3, for any $\tilde{\lambda} \in C_{\tilde{B}}$, $(\mathcal{L}_{\tilde{B}} - \tilde{\lambda}I)^{-1}$ exists and is everywhere densely defined. We need to show that $(\mathcal{L}_{\tilde{B}} - \tilde{\lambda}I)^{-1}$ is unbounded. For any $\tilde{\lambda} \in C_{\tilde{B}}$, there exists a root of $\tilde{f}_B(w, \tilde{\lambda})$ of modulus one,

$$w_* = e^{i\theta}, \quad \theta \in [0, 2\pi).$$

Define the elements

$$z_n^{(N)} = \begin{cases} e^{in\theta}, & n \leq N, \\ 0, & n > N. \end{cases}$$

Then $z^{(N)} \in \ell_2$ for each finite N , and $\|z^{(N)}\| \rightarrow \infty$, as $N \rightarrow \infty$. There exists a constant d independent of N , such that

$$\|(\tilde{B} - \tilde{\lambda}I)z^{(N)}\| \leq d, \quad \forall N.$$

Thus

$$\frac{\|z^{(N)}\|}{\|(\tilde{B} - \tilde{\lambda}I)z^{(N)}\|} \longrightarrow \infty, \quad \text{as } N \longrightarrow \infty.$$

Therefore, $(\mathcal{L}_{\tilde{B}} - \tilde{\lambda}I)^{-1}$ is unbounded, and $\tilde{\lambda} \in \sigma_c(\tilde{B})$. Next we show that if $\tilde{\lambda} \notin C_{\tilde{B}}$, then $\tilde{\lambda} \in \rho(\tilde{B})$. For any $\tilde{\lambda} \notin C_{\tilde{B}}$, the corresponding roots of $\tilde{f}_B(w, \tilde{\lambda})$ are (VI.19), such that

$$|w_*| = |-w_*| < 1 < |w_*^{-1}| = |(-w_*)^{-1}|. \quad (\text{VI.35})$$

For any $y \in \ell_2$, we want to construct a solution to

$$(\tilde{B} - \tilde{\lambda}I)z = y, \quad (\text{VI.36})$$

using the method of variation of coefficients. Explicitly, we need to solve

$$z_n - \tilde{\lambda} z_{n+2} + z_{n+4} = y_{n+2}, \quad (n \geq 1) \quad (\text{VI.37})$$

under the constraints

$$\begin{cases} -\tilde{\lambda}z_1 + z_2 + z_3 &= y_1, \\ z_1 - \tilde{\lambda}z_2 + z_4 &= y_2. \end{cases} \quad (\text{VI.38})$$

Assume a solution to (VI.37) has the form

$$z_n = c_n^{(1)} w_*^n + c_n^{(2)} (-w_*)^n + c_n^{(3)} w_*^{-n} + c_n^{(4)} (-w_*)^{-n} . \quad (\text{VI.39})$$

If

$$\begin{aligned} & \Delta c_n^{(1)} w_*^{n+1} + \Delta c_n^{(2)} (-w_*)^{n+1} \\ & + \Delta c_n^{(3)} w_*^{-(n+1)} + \Delta c_n^{(4)} (-w_*)^{-(n+1)} = 0 , \end{aligned} \quad (\text{VI.40})$$

$$\begin{aligned} & \Delta c_n^{(1)} w_*^{n+2} + \Delta c_n^{(2)} (-w_*)^{n+2} \\ & + \Delta c_n^{(3)} w_*^{-(n+2)} + \Delta c_n^{(4)} (-w_*)^{-(n+2)} = 0 , \end{aligned} \quad (\text{VI.41})$$

$$\begin{aligned} & \Delta c_n^{(1)} w_*^{n+3} + \Delta c_n^{(2)} (-w_*)^{n+3} \\ & + \Delta c_n^{(3)} w_*^{-(n+3)} + \Delta c_n^{(4)} (-w_*)^{-(n+3)} = 0 , \end{aligned} \quad (\text{VI.42})$$

$$\Delta c_n^{(1)} w_*^{n+4} + \Delta c_n^{(2)} (-w_*)^{n+4}$$

$$+\Delta c_n^{(3)} w_*^{-(n+4)} + \Delta c_n^{(4)} (-w_*)^{-(n+4)} = y_{n+2}; \quad (\text{VI.43})$$

where $\Delta c_n^{(\ell)} = c_{n+1}^{(\ell)} - c_n^{(\ell)}$, ($\ell = 1, 2, 3, 4$), then z_n given in (VI.39) solves (VI.37). Solving (VI.40–VI.43), we have

$$\Delta c_n^{(\ell)} = (-1)^\ell y_{n+2} \frac{D_n^{(\ell)}}{W_n}, \quad (\ell = 1, 2, 3, 4) \quad (\text{VI.44})$$

where

$$W_n = \begin{vmatrix} w_*^{n+1} & (-w_*)^{n+1} & w_*^{-(n+1)} & (-w_*)^{-(n+1)} \\ w_*^{n+2} & (-w_*)^{n+2} & w_*^{-(n+2)} & (-w_*)^{-(n+2)} \\ w_*^{n+3} & (-w_*)^{n+3} & w_*^{-(n+3)} & (-w_*)^{-(n+3)} \\ w_*^{n+4} & (-w_*)^{n+4} & w_*^{-(n+4)} & (-w_*)^{-(n+4)} \end{vmatrix}, \quad (\text{VI.45})$$

$$D_n^{(1)} = \begin{vmatrix} (-w_*)^{n+1} & w_*^{-(n+1)} & (-w_*)^{-(n+1)} \\ (-w_*)^{n+2} & w_*^{-(n+2)} & (-w_*)^{-(n+2)} \\ (-w_*)^{n+3} & w_*^{-(n+3)} & (-w_*)^{-(n+3)} \end{vmatrix}$$

$$= 2w_*^{-n} [w_*^{-4} - 1], \quad (\text{VI.46})$$

$$\begin{aligned}
D_n^{(2)} &= \begin{vmatrix} w_*^{n+1} & w_*^{-(n+1)} & (-w_*)^{-(n+1)} \\ w_*^{n+2} & w_*^{-(n+2)} & (-w_*)^{-(n+2)} \\ w_*^{n+3} & w_*^{-(n+3)} & (-w_*)^{-(n+3)} \end{vmatrix} \\
&= 2(-w_*)^{-n} \left[1 - w_*^{-4} \right] , \tag{VI.47}
\end{aligned}$$

$$\begin{aligned}
D_n^{(3)} &= \begin{vmatrix} w_*^{n+1} & (-w_*)^{n+1} & (-w_*)^{-(n+1)} \\ w_*^{n+2} & (-w_*)^{n+2} & (-w_*)^{-(n+2)} \\ w_*^{n+3} & (-w_*)^{n+3} & (-w_*)^{-(n+3)} \end{vmatrix} \\
&= 2w_*^n \left[w_*^4 - 1 \right] , \tag{VI.48}
\end{aligned}$$

$$\begin{aligned}
D_n^{(4)} &= \begin{vmatrix} w_*^{n+1} & (-w_*)^{n+1} & w_*^{-(n+1)} \\ w_*^{n+2} & (-w_*)^{n+2} & w_*^{-(n+2)} \\ w_*^{n+3} & (-w_*)^{n+3} & w_*^{-(n+3)} \end{vmatrix} \\
&= 2(-w_*)^n \left[1 - w_*^4 \right] . \tag{VI.49}
\end{aligned}$$

The W_n defined in (VI.45) satisfies the Wronskian relation

$$W_{n+1} = W_n . \quad (\text{VI.50})$$

The representation (VI.44) can be extended to $n \geq 0$. Choose $c_0^{(\ell)} = 0$, we have

$$c_n^{(\ell)} = \sum_{j=0}^{n-1} \Delta c_j^{(\ell)} = \frac{1}{W_0} \sum_{j=0}^{n-1} D_j^{(\ell)} (-1)^\ell y_{j+2} , \quad (n \geq 1) . \quad (\text{VI.51})$$

The expressions (VI.46–VI.49) lead to

$$c_n^{(1)} = \frac{2 [1 - w_*^{-4}]}{W_0} \sum_{j=0}^{n-1} w_*^{-j} y_{j+2} , \quad (\text{VI.52})$$

$$c_n^{(2)} = \frac{2 [1 - w_*^{-4}]}{W_0} \sum_{j=0}^{n-1} (-w_*)^{-j} y_{j+2} , \quad (\text{VI.53})$$

$$c_n^{(3)} = \frac{2 [1 - w_*^4]}{W_0} \sum_{j=0}^{n-1} w_*^j y_{j+2} , \quad (\text{VI.54})$$

$$c_n^{(4)} = \frac{2 [1 - w_*^4]}{W_0} \sum_{j=0}^{n-1} (-w_*)^j y_{j+2} . \quad (\text{VI.55})$$

With these representations of $c_n^{(\ell)}$, z_n given by (VI.39) is a special solution to (VI.37). The general solution to (VI.37) is

$$z_n = \left(c_n^{(1)} + a^{(1)} \right) w_*^n + \left(c_n^{(2)} + a^{(2)} \right) (-w_*)^n$$

$$+ \left(c_n^{(3)} + a^{(3)} \right) w_*^{-n} + \left(c_n^{(4)} + a^{(4)} \right) (-w_*)^{-n}, \quad (\text{VI.56})$$

where $a^{(\ell)}$ ($\ell = 1, 2, 3, 4$) are arbitrary constants. Set

$$a^{(3)} = - \frac{2 [1 - w_*^4]}{W_0} \sum_{j=0}^{\infty} w_*^j y_{j+2}, \quad (\text{VI.57})$$

$$a^{(4)} = - \frac{2 [1 - w_*^4]}{W_0} \sum_{j=0}^{\infty} (-w_*)^j y_{j+2}. \quad (\text{VI.58})$$

Let

$$f_n = c_n^{(1)} w_*^n + c_n^{(2)} (-w_*)^n + (c_n^{(3)} + a^{(3)}) w_*^{-n} + (c_n^{(4)} + a^{(4)}) (-w_*)^{-n}; \quad (\text{VI.59})$$

then from expressions (VI.52–VI.55) and (VI.57–VI.59), we have

$$\begin{aligned} f_n &= \frac{2 [1 - w_*^4]}{W_0} \sum_{j=0}^{n-1} \left[w_*^{n-j} + (-w_*)^{n-j} \right] y_{j+2} \\ &\quad - \frac{2 [1 - w_*^4]}{W_0} \sum_{j=n}^{\infty} \left[w_*^{j-n} + (-w_*)^{j-n} \right] y_{j+2}. \end{aligned} \quad (\text{VI.60})$$

Finally,

$$z_n = a^{(1)} w_*^n + a^{(2)} (-w_*)^n + f_n. \quad (\text{VI.61})$$

Next we choose $a^{(1)}$ and $a^{(2)}$ to satisfy the constraints (VI.38):

$$M \begin{pmatrix} a^{(1)} \\ a^{(2)} \end{pmatrix} = \begin{bmatrix} y_1 + \tilde{\lambda}f_1 - f_2 - f_3 \\ y_2 - f_1 + \tilde{\lambda}f_2 - f_4 \end{bmatrix}, \quad (\text{VI.62})$$

where

$$M = \begin{bmatrix} -\tilde{\lambda}w_* + w_*^2 + w_*^3 & \tilde{\lambda}w_* + w_*^2 - w_*^3 \\ w_* - \tilde{\lambda}w_*^2 + w_*^4 & -w_* - \tilde{\lambda}w_*^2 + w_*^4 \end{bmatrix}.$$

As shown in the proof of Theorem VI.2, M is nonsingular. Then,

$$\begin{pmatrix} a^{(1)} \\ a^{(2)} \end{pmatrix} = M^{-1} \begin{bmatrix} y_1 + \tilde{\lambda}f_1 - f_2 - f_3 \\ y_2 - f_1 + \tilde{\lambda}f_2 - f_4 \end{bmatrix}, \quad (\text{VI.63})$$

where

$$M^{-1} = \tilde{\kappa}^{-1} \begin{bmatrix} -w_* - \tilde{\lambda}w_*^2 + w_*^4 & -\tilde{\lambda}w_* - w_*^2 + w_*^3 \\ -w_* + \tilde{\lambda}w_*^2 - w_*^4 & -\tilde{\lambda}w_* + w_*^2 + w_*^3 \end{bmatrix},$$

where $\tilde{\kappa} = 2w_*^3(w_*^4 - 2\tilde{\lambda}w_*^2 + \tilde{\lambda}^2 - 1)$. Thus

$$z_n = \left(w_*^n, (-w_*)^n \right) M^{-1} \begin{bmatrix} y_1 + \tilde{\lambda}f_1 - f_2 - f_3 \\ y_2 - f_1 + \tilde{\lambda}f_2 - f_4 \end{bmatrix} + f_n \quad (\text{VI.64})$$

solves Equation (VI.36). Rewrite f_n given in (VI.60) as follows:

$$f_n = \sum_{j=1}^{\infty} g(n, j) y_j, \quad (\text{VI.65})$$

where

$$g(n, j) = \begin{cases} 0, & j = 1; \\ \frac{2[1-w_*^{-4}]}{W_0} [w_*^{n-j+2} + (-w_*)^{n-j+2}], & 2 \leq j \leq n+1; \\ -\frac{2[1-w_*^4]}{W_0} [w_*^{j-n-2} + (-w_*)^{j-n-2}], & j \geq n+2. \end{cases}$$

Rewrite z_n given in (VI.64) as follows:

$$z_n = \sum_{j=1}^{\infty} G(n, j) y_j, \quad (\text{VI.66})$$

where

$$G(n, j) = \begin{pmatrix} w_*^n, & (-w_*)^n \end{pmatrix} M^{-1} \begin{pmatrix} \delta_{1,j} + \tilde{\lambda} g(1, j) - g(2, j) - g(3, j) \\ \delta_{2,j} - g(1, j) + \tilde{\lambda} g(2, j) - g(4, j) \end{pmatrix} + g(n, j), \quad (\text{VI.67})$$

where $\delta_{\ell,j}$ is the Kronecker delta: $\delta_{\ell,j} = 1$ ($\ell = j$), $\delta_{\ell,j} = 0$ ($\ell \neq j$). From the expression (VI.67), we see that there exists a constant K independent of n, j ; such that

$$\sum_{j=1}^{\infty} |G(n, j)| \leq K, \quad \forall n = 1, 2, \dots; \quad (\text{VI.68})$$

$$\sum_{n=1}^{\infty} |G(n, j)| \leq K, \quad \forall j = 1, 2, \dots. \quad (\text{VI.69})$$

Then, we have the ℓ_{∞} norm relation,

$$\begin{aligned} \|z\|_{\infty} = \sup_n |z_n| &\leq \sup_n \sum_{j=1}^{\infty} |G(n, j)| |y_j| \\ &\leq \left[\sup_n \sum_{j=1}^{\infty} |G(n, j)| \right] \|y\|_{\infty} \\ &\leq K \|y\|_{\infty}, \end{aligned} \quad (\text{VI.70})$$

and the ℓ_1 norm relation,

$$\begin{aligned} \|z\|_1 = \lim_{N \rightarrow \infty} \left[\sum_{n=1}^N |z_n| \right] &\leq \lim_{N \rightarrow \infty} \left[\sum_{n=1}^N \sum_{j=1}^{\infty} |G(n, j)| |y_j| \right] \\ &= \lim_{N \rightarrow \infty} \left[\sum_{j=1}^{\infty} |y_j| \sum_{n=1}^N |G(n, j)| \right] \\ &\leq K \sum_{j=1}^{\infty} |y_j| = K \|y\|_1. \end{aligned} \quad (\text{VI.71})$$

Thus the linear operator defined in (VI.66) which maps y into z , is bounded in ℓ_{∞} and ℓ_1 . Therefore, by Riesz convexity theorem [23] [24] [18], $(\mathcal{L}_{\tilde{B}} - \tilde{\lambda}I)^{-1}$ defined in (VI.66) is

bounded in ℓ_2 . Since by Theorems VI.2 and VI.3, $(\mathcal{L}_{\tilde{B}} - \tilde{\lambda}I)^{-1}$ exists and is everywhere densely defined and is bounded, we have $\tilde{\lambda} \in \rho(\tilde{B})$. In summary, we have shown that if $\tilde{\lambda} \in C_{\tilde{B}}$, then $\tilde{\lambda} \in \sigma_c(\tilde{B})$; and if $\tilde{\lambda} \notin C_{\tilde{B}}$, then $\tilde{\lambda} \in \rho(\tilde{B})$; thus, $\sigma_c(\tilde{B}) = C_{\tilde{B}}$ and $\rho(\tilde{B}) = (C_{\tilde{B}})'$. Equivalently, $\sigma_c(B) = C_B$ and $\rho(B) = (C_B)'$. \square

In summary, the spectrum of \mathcal{L}_B is as depicted in Figure 5.

VI.3. The Spectra of the Linear Operator \mathcal{L}_A

Now we apply Weyl's essential spectrum theorem [19] to obtain the spectral theorem for \mathcal{L}_A .

Theorem VI.5 (The Spectral Theorem of \mathcal{L}_A) 1. If $\Sigma_{\hat{k}} \cap \bar{D}_{|p|} = \emptyset$, then the entire

ℓ_2 spectrum of the linear operator \mathcal{L}_A is its continuous spectrum which is the spectral curve C_B defined in (VI.33), i.e. $\sigma(\mathcal{L}_A) = \sigma_c(\mathcal{L}_A) = C_B$. See Figure 5.

2. If $\Sigma_{\hat{k}} \cap \bar{D}_{|p|} \neq \emptyset$, then the entire essential ℓ_2 spectrum of the linear operator \mathcal{L}_A is its continuous spectrum which is the spectral curve C_B defined in (VI.33), i.e. $\sigma_{ess}(\mathcal{L}_A) = \sigma_c(\mathcal{L}_A) = C_B$. That is, the residual spectrum of \mathcal{L}_A is empty, $\sigma_r(\mathcal{L}_A) = \emptyset$. The point spectrum of \mathcal{L}_A is symmetric with respect to both real and imaginary axes. See Figure 6.

Proof: First, we want to show that in both cases, the residual spectrum of \mathcal{L}_A is empty. By Weyl's essential spectrum theorem [19], the essential spectrum of $i\mathcal{L}_A$ is the same with the essential spectrum of $i\mathcal{L}_B$, $\sigma_{ess}(i\mathcal{L}_A) = \sigma_{ess}(i\mathcal{L}_B) = iC_B$. Let $i\lambda_r \in \sigma_r(i\mathcal{L}_A)$, then $i\lambda_r \in iC_B$. By the argument in the proof of Theorem VI.3, $i\lambda_r \in \sigma_p((i\mathcal{L}_A)^*)$, where $(i\mathcal{L}_A)^*$ is the adjoint of $i\mathcal{L}_A$,

$$(i\mathcal{L}_A)^* = i\mathcal{L}_B + (i\mathcal{L}_C)^* .$$

By Weyl's essential spectrum theorem [19], the essential spectrum of $(i\mathcal{L}_A)^*$ is the same with the essential spectrum of $i\mathcal{L}_B$, $\sigma_{\text{ess}}((i\mathcal{L}_A)^*) = \sigma_{\text{ess}}(i\mathcal{L}_B) = iC_B$. Thus, $i\lambda_r \in \sigma_{\text{ess}}((i\mathcal{L}_A)^*)$. Since $\sigma_{\text{ess}}((i\mathcal{L}_A)^*)$ and $\sigma_p((i\mathcal{L}_A)^*)$ are disjoint, $\sigma_r(i\mathcal{L}_A) = \emptyset$. The claim $\sigma_p(\mathcal{L}_A) = \emptyset$ in case 1 follows from the proof of Lemma V.2 and the fact that the spectral curve C_B corresponds to $\text{Re}\{\tilde{a}\} = 0$ and $|\tilde{a}| \leq 2$. The property of $\sigma_p(\mathcal{L}_A)$ in case 2 has been proved in Theorem IV.1. Then Weyl's essential spectrum theorem implies the rest of the claims. \square

Remark VI.2 *By the above theorem, the computation of eigenvalues is reduced to the case that $\Sigma_{\hat{k}} \cap \bar{D}_{|p|} \neq \emptyset$. By Corollary 1, if $\lambda \notin C_B = \sigma_{\text{ess}}(\mathcal{L}_A)$, the two continued fractions (V.4) and (V.6) converge. And solutions of equation (V.17) lead to eigenvalues.*

Remark VI.3 *The width of the continuous spectrum $\sigma_c(\mathcal{L}_A)$ is $4|b|$, where $b = -a|p|^{-2}$ and $a = \frac{1}{2} |\Gamma| \begin{vmatrix} p_1 & \hat{k}_1 \\ p_2 & \hat{k}_2 \end{vmatrix}$. Although $|a|$ can increase to infinity as $|k|$ increases to infinity, a is essentially a scaling factor for \mathcal{L}_A as can be seen in the expression for the infinite-matrix A .*

Next we discuss an alternative way of representing eigenvalues. This approach is not useful for practical computation. Consider the linear difference equation,

$$(A - \lambda I) z = 0, \quad (\text{VI.72})$$

where A defined in (VI.6) is the representation matrix of \mathcal{L}_A . Explicitly,

$$\begin{cases} \rho_{n-1} z_{2(n-1)} - \hat{\lambda} z_{2n} + \rho_{n+1} z_{2(n+1)} = 0, & (n \geq 2), \\ \rho_{-n+1} z_{2(n-1)+1} - \hat{\lambda} z_{2n+1} + \rho_{-n-1} z_{2(n+1)+1} = 0, & (n \geq 1), \end{cases} \quad (\text{VI.73})$$

under the constraints

$$\begin{cases} -\hat{\lambda} z_1 + \rho_1 z_2 + \rho_{-1} z_3 = 0, \\ \rho_0 z_1 - \hat{\lambda} z_2 + \rho_2 z_4 = 0, \end{cases} \quad (\text{VI.74})$$

where $\hat{\lambda} = (ia)^{-1} \lambda$. By Theorem VI.1, the linear operator \mathcal{L}_A is a compact perturbation of \mathcal{L}_B . Thus, the difference equation (VI.73) is of Poincaré-Perron type. The Poincaré-Perron theorem stated specifically for the difference equation (VI.72) is as follows [16], [17], [18]:

Theorem VI.6 (Poincaré-Perron Theorem) *For any $\lambda \in C$, let w_* , $-w_*$, $\frac{1}{w_*}$, $-\frac{1}{w_*}$ be the roots of the characteristic polynomial $f_B(w, \lambda)$ defined in (VI.16), which are given in (VI.19), where $|w_*| \leq 1$. Then there exists a fundamental set of solutions $z_n^{(j)}$ ($j = 1, 2, 3, 4$) to the difference equation (VI.73), such that*

$$\limsup_{n \rightarrow \infty} \left| z_n^{(j)} \right|^{\frac{1}{n}} = |w_*|, \quad (j = 1, 2), \quad (\text{VI.75})$$

$$\limsup_{n \rightarrow \infty} \left| z_n^{(j)} \right|^{\frac{1}{n}} = \frac{1}{|w_*|}, \quad (j = 3, 4).$$

It is easy to see that

$$z^{(j)} \in \ell_2, \quad (j = 1, 2); \quad z^{(j)} \notin \ell_2, \quad (j = 3, 4);$$

if $|w_*| < 1$. By definition, when $\lambda \notin C_B$ (defined in (VI.33)), $|w_*| < 1$. Next we study the conditions for the point spectrum of \mathcal{L}_A . Let

$$z_n = c_1 z_n^{(1)} + c_2 z_n^{(2)}. \quad (\text{VI.76})$$

Substitute z_n into the constraints (VI.74), we have

$$M \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0, \quad (\text{VI.77})$$

where

$$M = \begin{pmatrix} -\hat{\lambda}z_1^{(1)} + \rho_1z_2^{(1)} + \rho_{-1}z_3^{(1)} & -\hat{\lambda}z_1^{(2)} + \rho_1z_2^{(2)} + \rho_{-1}z_3^{(2)} \\ \rho_0z_1^{(1)} - \hat{\lambda}z_2^{(1)} + \rho_2z_4^{(1)} & \rho_0z_1^{(2)} - \hat{\lambda}z_2^{(2)} + \rho_2z_4^{(2)} \end{pmatrix}. \quad (\text{VI.78})$$

Theorem VI.7 *If $\lambda \notin C_B$ (the spectral curve for \mathcal{L}_B , defined in (VI.33)), and $\det M = 0$ (where M is defined in (VI.78)), then $\lambda \in \sigma_p(A)$ (the point spectrum of \mathcal{L}_A).*

Proof: If $\det M = 0$, then there is a nontrivial solution to (VI.77). Thus there is a nonzero solution to (VI.73), which satisfies the constraints (VI.74). Therefore, λ is an eigenvalue. \square

VII. Conclusion

In this paper, we study the linearized two-dimensional Euler equation at a stationary state. This equation decouples into infinite many invariant subsystems. Each invariant subsystem is shown to be a linear Hamiltonian system of infinite dimensions. Another important invariant besides the Hamiltonian for each invariant subsystem is found, and is utilized to prove an “unstable disk theorem” through a simple Energy-Casimir argument. The eigenvalues of the linear Hamiltonian system are of four types: real pairs $(c, -c)$, purely imaginary pairs $(id, -id)$, quadruples $(\pm c \pm id)$, and zero eigenvalues. The eigenvalues are studied through continued fractions. The spectral equation for each invariant subsystem is a Poincaré-type difference equation, i.e. it can be represented as the spectral equation of an infinite matrix operator, and the infinite matrix operator is a sum of a constant-coefficient infinite matrix operator and a compact infinite matrix operator. We have a complete spectral theory. The essential spectrum of each invariant subsystem is a bounded band of continuous spectrum. The point spectrum can be computed through continued fractions.

This study is the first step toward understanding the unstable manifold structures of stationary states of the two-dimensional Euler equation, which we believe to be the key for understanding two-dimensional turbulence. In particular, we will be interested in investigating whether or not the unstable manifolds of 2D Euler equations are degenerate (i.e. figure eight structures). Degeneracy will imply that the dynamics of 2D Euler equations is not turbulent.

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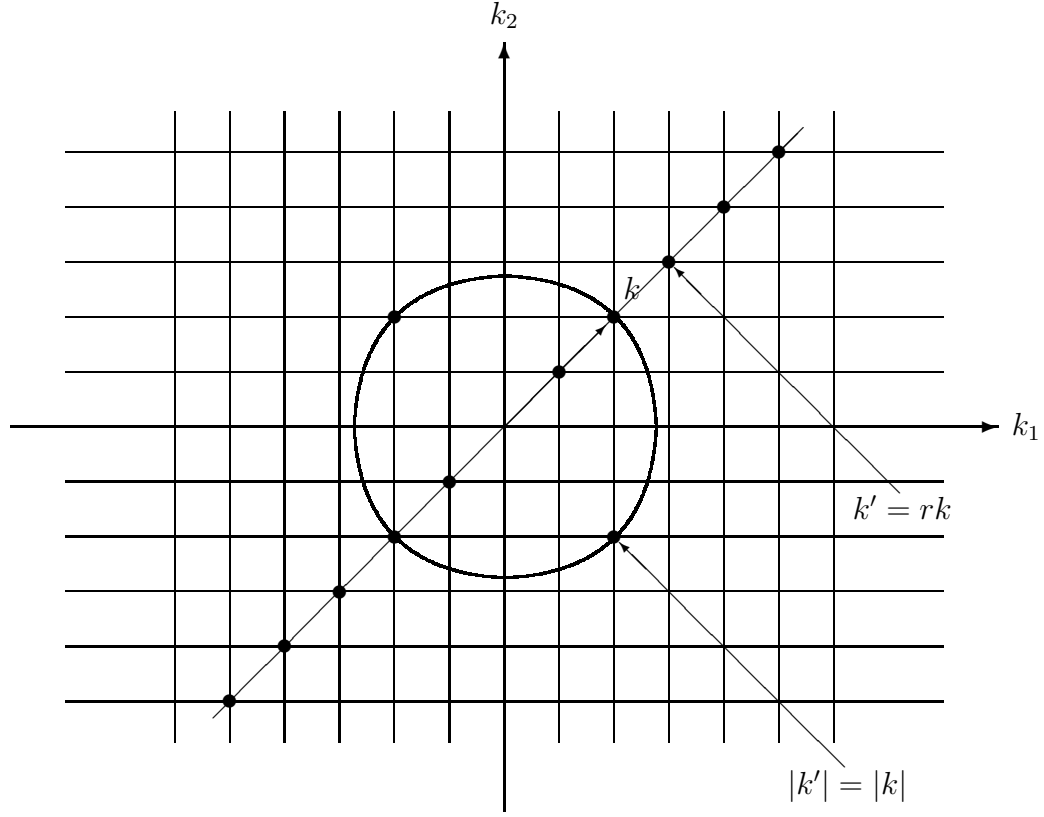


Figure 1: An illustration on the locations of the modes $(k' = rk)$ and $(|k'| = |k|)$ in the definitions of E_k^1 and E_k^2 (Proposition 1).

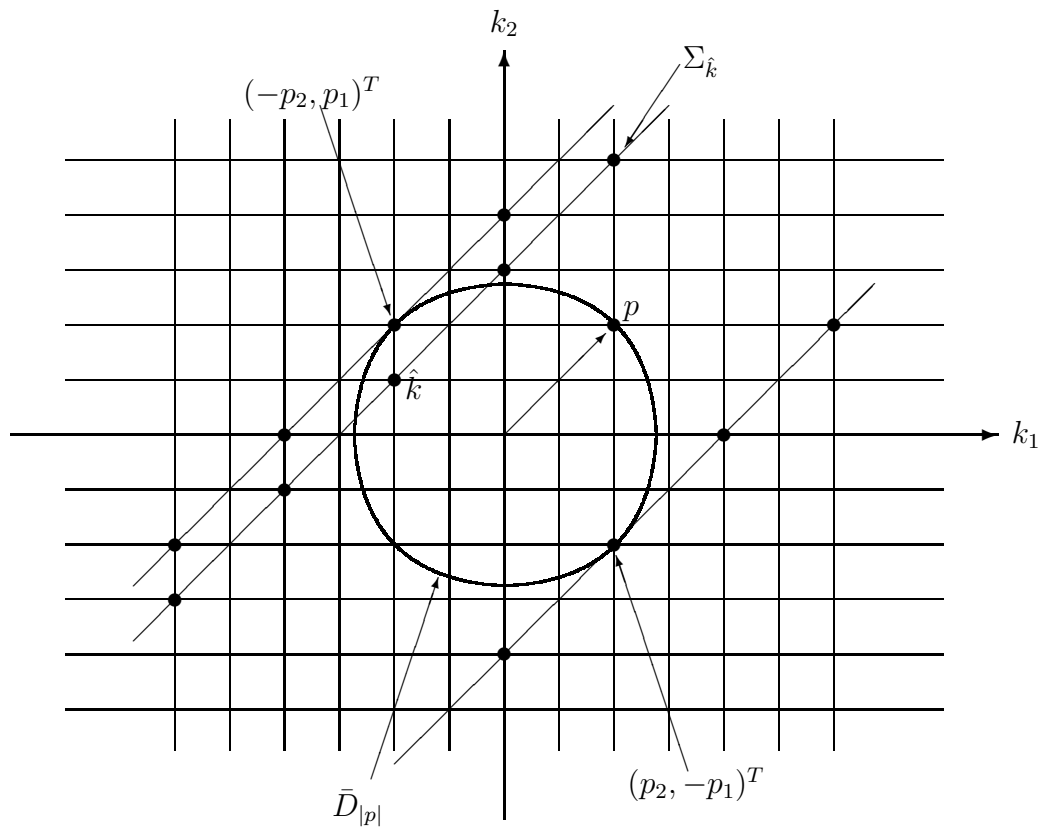


Figure 2: An illustration of the classes $\Sigma_{\hat{k}}$ and the disk $\bar{D}_{|p|}$.

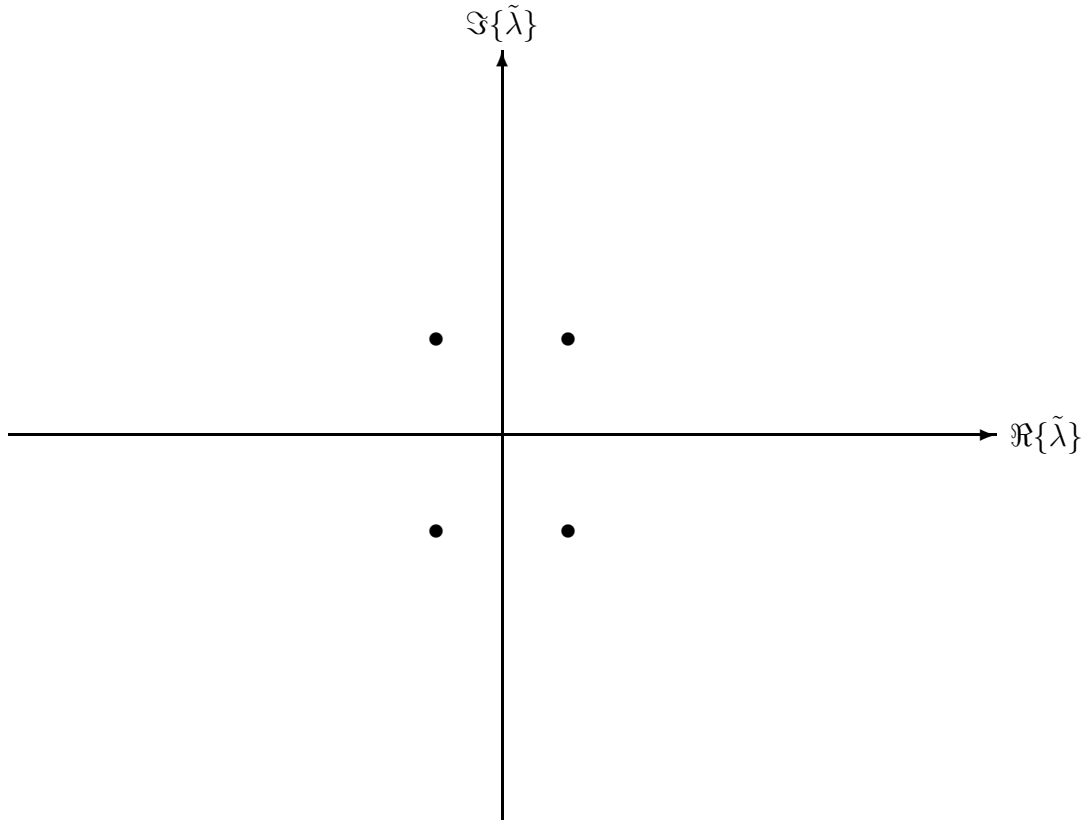


Figure 3: The quadruple of eigenvalues determined by equation (V.17) for the system led by the class $\Sigma_{\hat{k}}$ labeled by $\hat{k} = (1, 0)^T$, when $p = (1, 1)^T$.

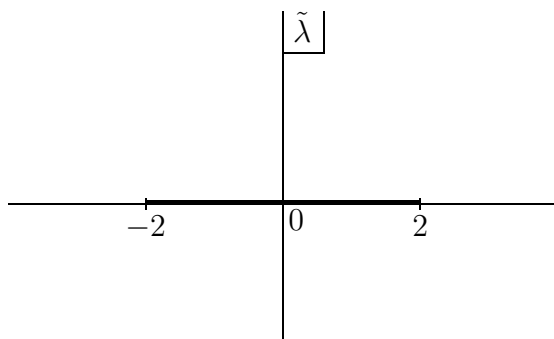


Figure 4: The spectral curve $C_{\tilde{B}}$.

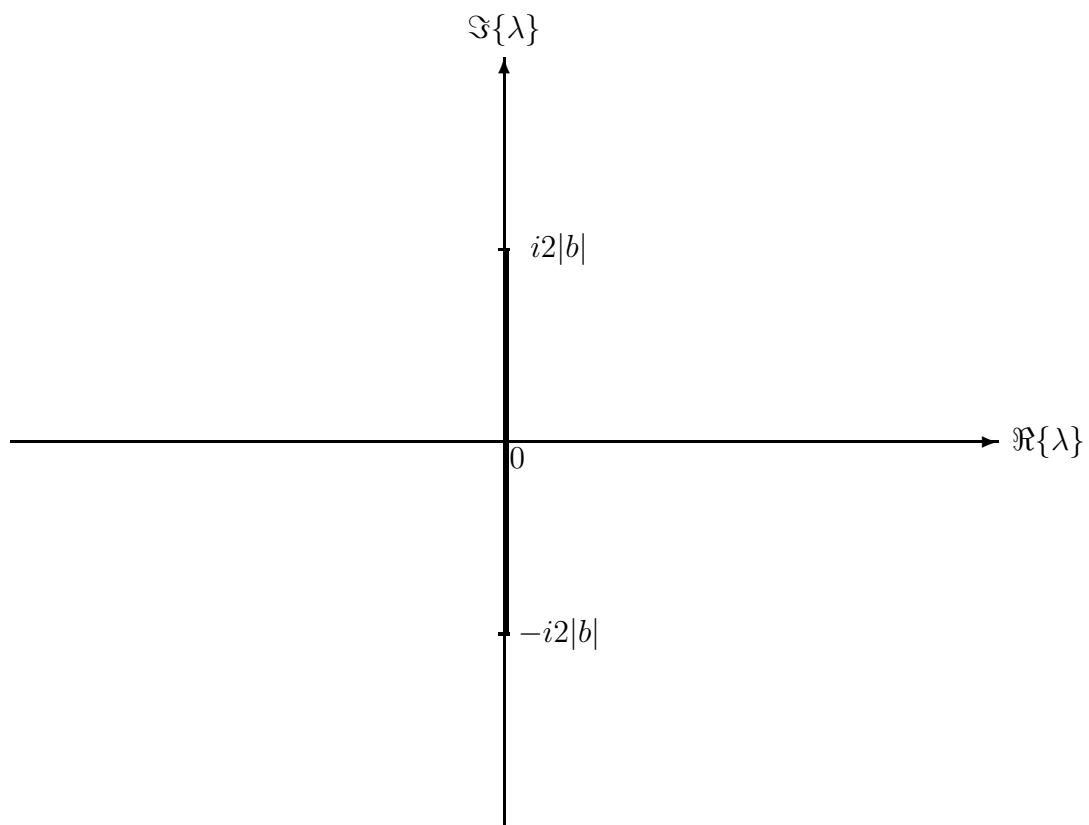


Figure 5: The continuous spectrum of \mathcal{L}_B and \mathcal{L}_A .

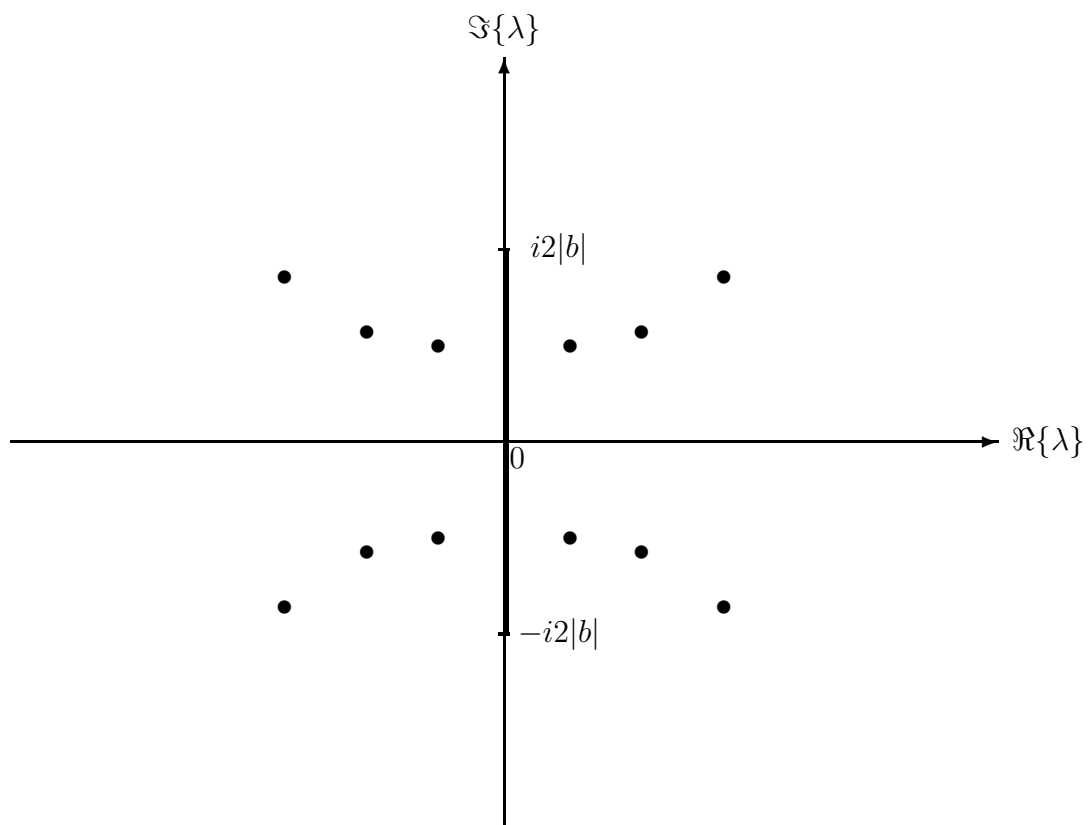


Figure 6: The spectrum of \mathcal{L}_A in case 2.

Figure 1. Caption: An illustration on the locations of the modes $(k' = rk)$ and $(|k'| = |k|)$ in the definitions of E_k^1 and E_k^2 (Proposition 1).

Figure 2. Caption: An illustration of the classes $\Sigma_{\hat{k}}$ and the disk $\bar{D}_{|p|}$.

Figure 3. Caption: The quadruple of eigenvalues determined by equation (V.17) for the system led by the class $\Sigma_{\hat{k}}$ labeled by $\hat{k} = (1, 0)^T$, when $p = (1, 1)^T$.

Figure 4. Caption: The spectral curve $C_{\tilde{B}}$.

Figure 5. Caption: The continuous spectrum of \mathcal{L}_B and \mathcal{L}_A .

Figure 6. Caption: The spectrum of \mathcal{L}_A in case 2.